Hyperbolic Geometry

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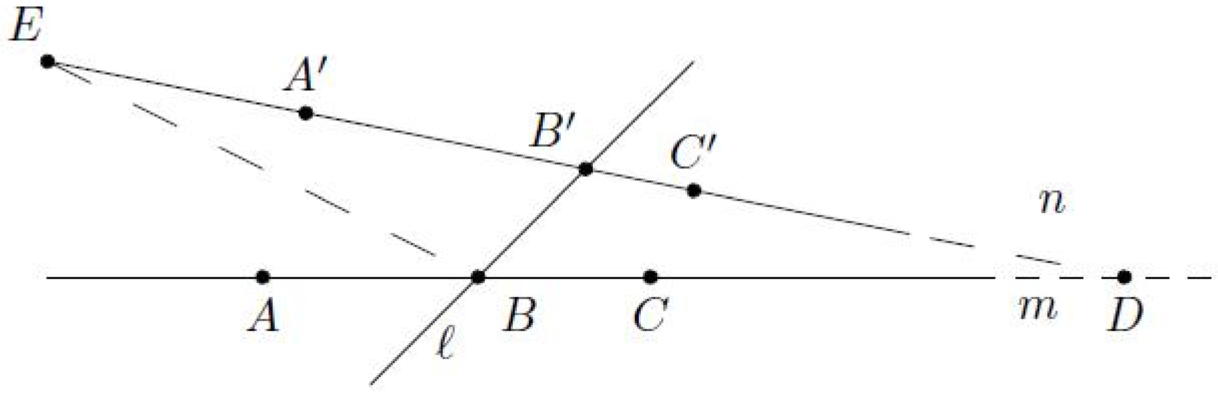
Part I
Axioms of Hyperbolic Geometry

- Axiom 1: We can draw a unique line segment between any two points.
- Axiom 2: Any line segment may be continued indefinitely.
- Axiom 3: A circle of any radius and any center can be drawn.
- Axiom 4: Any two right angles are congruent.
- Axiom 6: Given any two points P and Q, there exists an isometry f such that f(P) = Q. (translation)
- Axiom 7: Given a point P and any two points Q and R which are equidistant from P, there exists an isometry which fixes P and sends Q to R. (rotation)
- Axiom 8: Given any line l, there exists a map which fixes every point in l and fixes no other points. (reflection)
- Axiom 5H: Given any line l and any point P not on l, there exist two distinct lines l1 and l2 through P which do not intersect l. (parallels)

Thm 1: (Alternate Interior Angle Thm)

- Alternate interior angle congruent implies parallel.



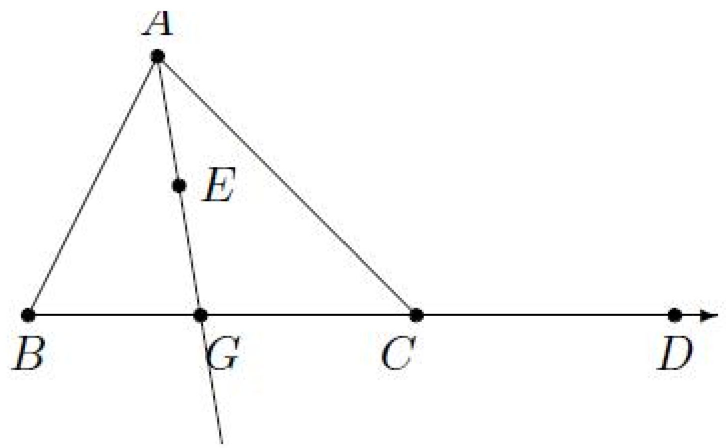
- Given ∠EB'B= ∠DBB'

Thm 2: weak exterior angle thm

- An exterior angle of a triangle is greater than either remote interior angle.

Claim: $\angle$ACD $>$ $\angle$BAC (or A

By symmetry)



Some corollaries

- *i) Every angle has a unique bisector.*
- *ii) Every segment has a unique perpendicular bisector.*
- *iii)In a triangle* $\triangle ABC$ *the greater angle lies opposite the greater side and the greater side lies opposite the greater angle;* i.e., $AB > BC$ *if and only if* $\angle C > \angle A$. (mentioned later by using hyperbolic law of sine)

Then we can define the degree measure to each angle

i) $\angle A$ *is a real number such that* $0 < \angle A < 180°$.

ii) $\angle A = 90°$ *if and only if* $\angle A$ *is a right angle.*

iii) $\angle A = \angle B$ *if and only if* $\angle A \cong \angle B$.

iv) If the ray AC *is interior to* $\angle DAB$, *then* $\angle DAB = \angle DAC + \angle CAB$.

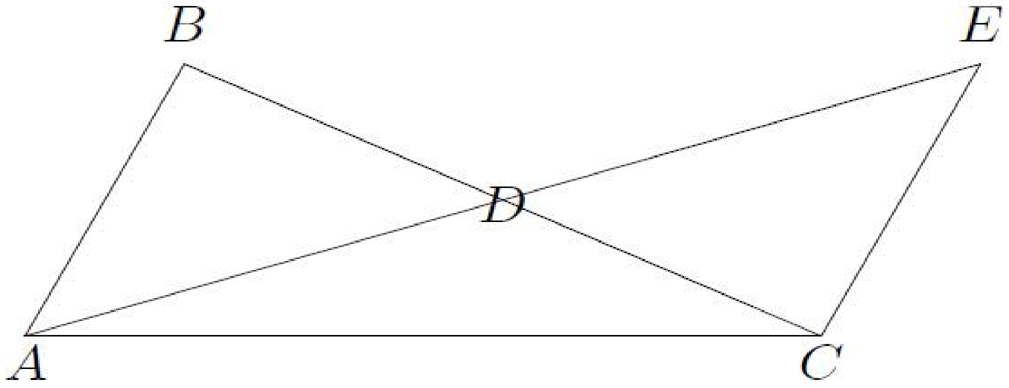
v) For every real number x *between 0 and 180, there exists an angle* $\angle A$ *such that* $\angle A = x°$.

vi) If $\angle B$ *is supplementary to* $\angle A$, *then* $\angle A + \angle B = 180\circ$.

Thm 3: (Saccheri-Lengendre)

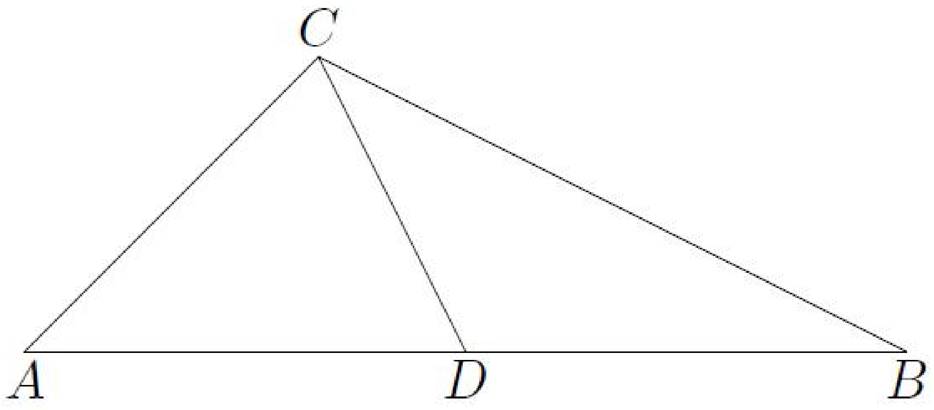
- *The sum of the degree measures of the three angles in any triangle is* less than or equal to 180 ˚： ∠*A* + ∠ *B* + ∠ *C* ≤180 ˚*:* there is an $x \in R^+$ so that

$$\angle A + \angle B + \angle C = 180^\circ + x.$$



Definition: $defect(\triangle ABC) = 180^\circ - (\angle A + \angle B + \angle C).$

Theorem 4.5 (Additivity of Defect) *Let $\triangle ABC$ be any triangle and let D be a point between A and B. Then* $\text{defect}(\triangle ABC) = \text{defect}(\triangle ACD) + \text{defect}(\triangle BCD)$.



Thm 5

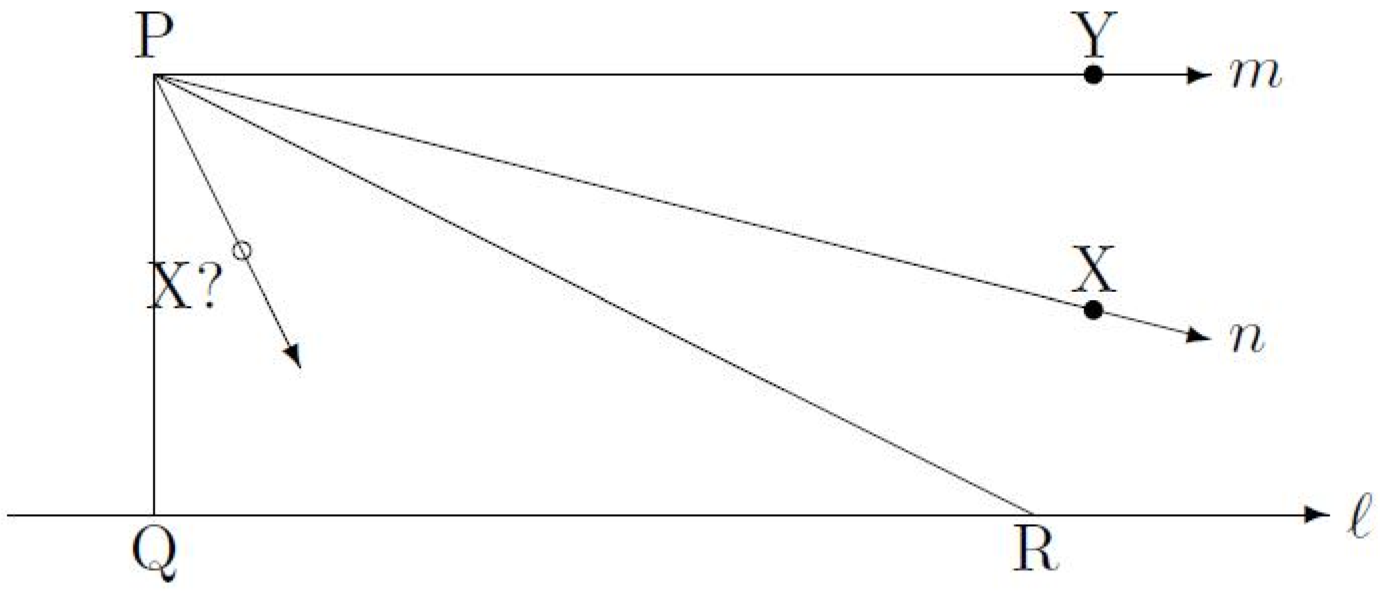
- *If there exists a triangle of defect 0, then a rectangle exists. If a rectangle exists, then every triangle has defect 0.*

This is a powerful theorem!(why)

- Steps to prove:
- 1. Construct a *right* triangle having defect 0.
- 2. From a right triangle of defect 0, construct a rectangle.
- 3. From one rectangle, construct arbitrarily large rectangles.
- 4. Prove that all *right* triangles have defect 0.
- 5. If every *right* triangle has defect 0, then *every* triangle has defect 0.

Thm 6 (property of hyperbolic geometry)

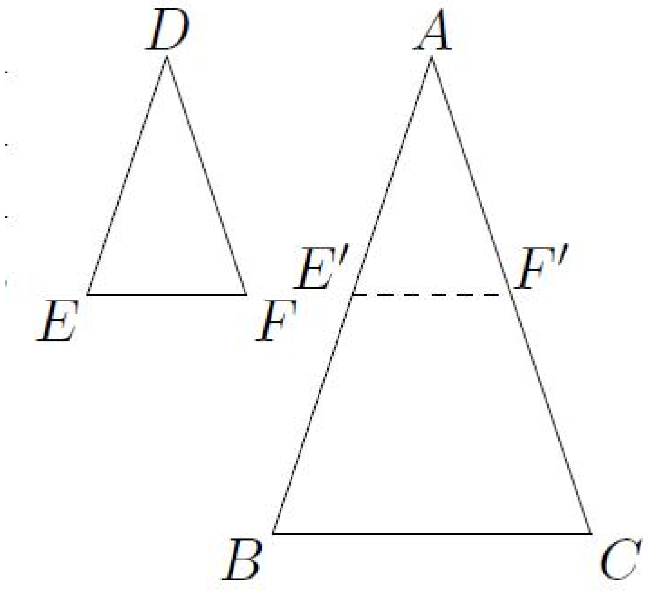
- *There exists a triangle whose angle sum is less than* 180 °.



- Then we have an idea to construct another parallel!

Thm 7(AAA Criterion)

- *In H^2 if ∠A = ∠ D, ∠ B = ∠ E, and ∠ C = ∠ F, then △ABC ≅ △ DEF. That is, if two triangles are* similar*, then they are* cong



Part 2 2.1Poincaré Half-plane Model

Let $\mathscr{H} = \{x + iy \mid y > 0\}$ together with the arclength element

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

So the Poincare arclength is

$$s_P = \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y} \, dt.$$

Isometries

-

$$(u(x,y), v(x,y))$$

is an isometry if

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + dy^2}{y^2}.$$

3 types of Isometries we will consider

- Horizontal translation:

$$T_a(x, y) = (u, v) = (x + a, y).$$

- Reflection by a verticle line:

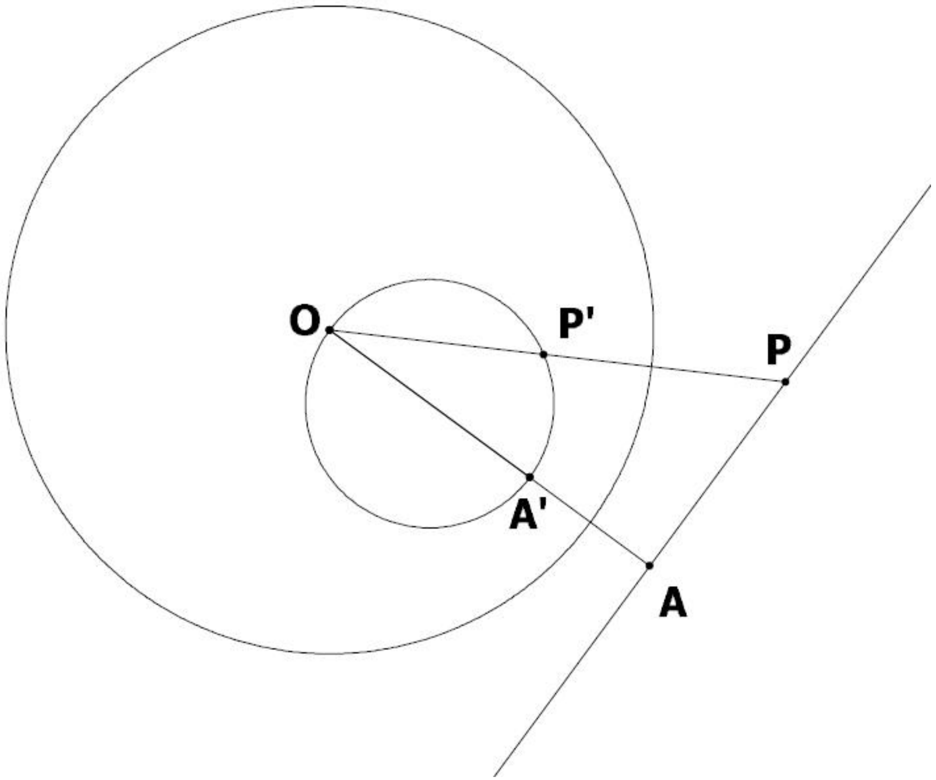
$$R_b(x, y) = (u, v) = (2b - x, y).$$

- Inversion in the unit circle:

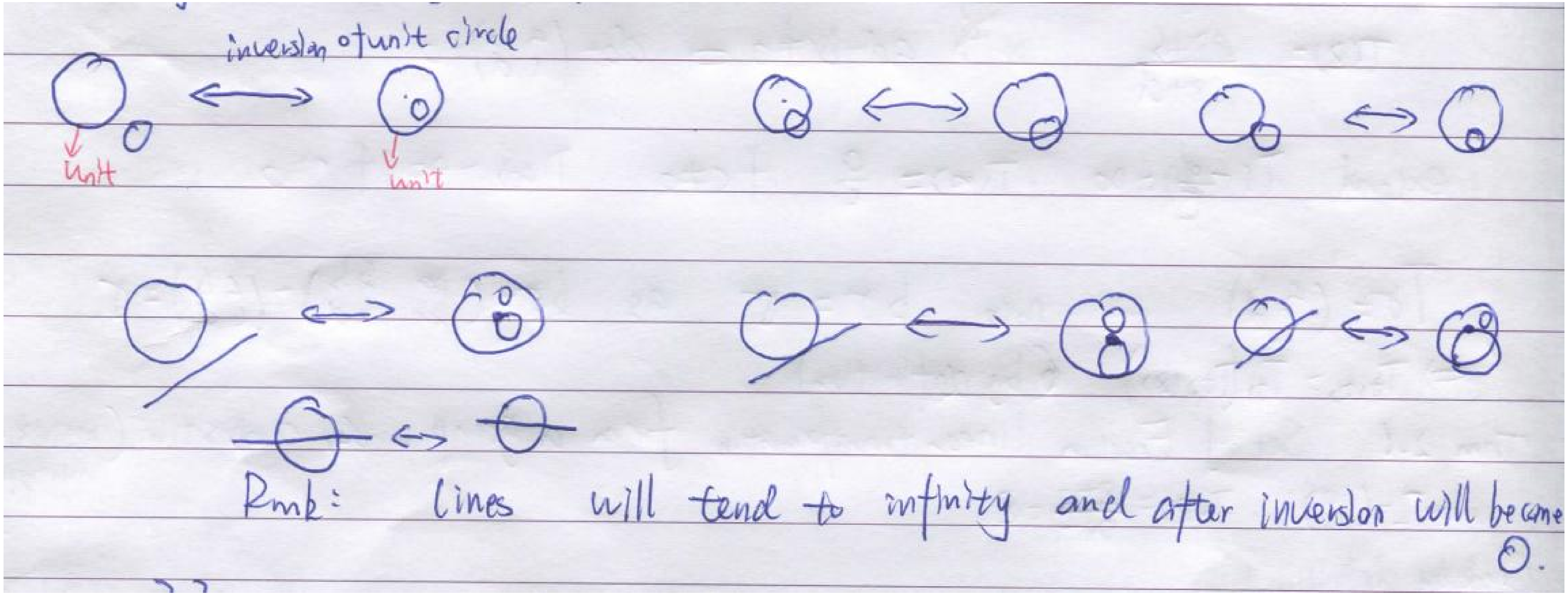
$$\Phi(x, y) = (u, v) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right).$$

Lemma 2.1

The image of a line which does not go through the origin O under inversion in the unit circle is a circle which goes through the origin O

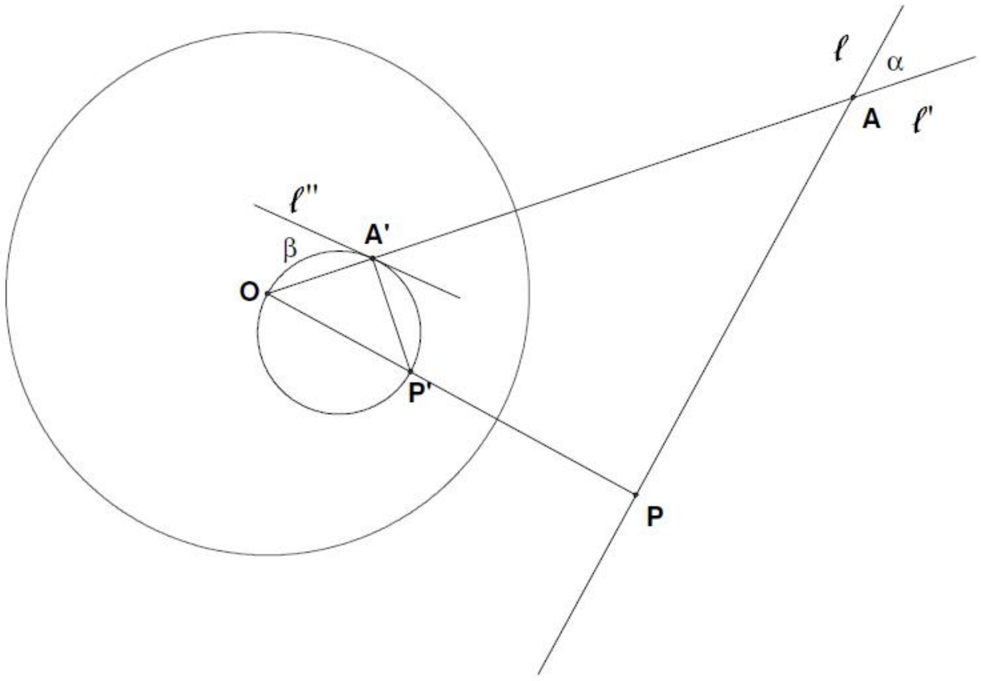


- Similarly, we could show other circles and lines under inversion are also lines or circles.



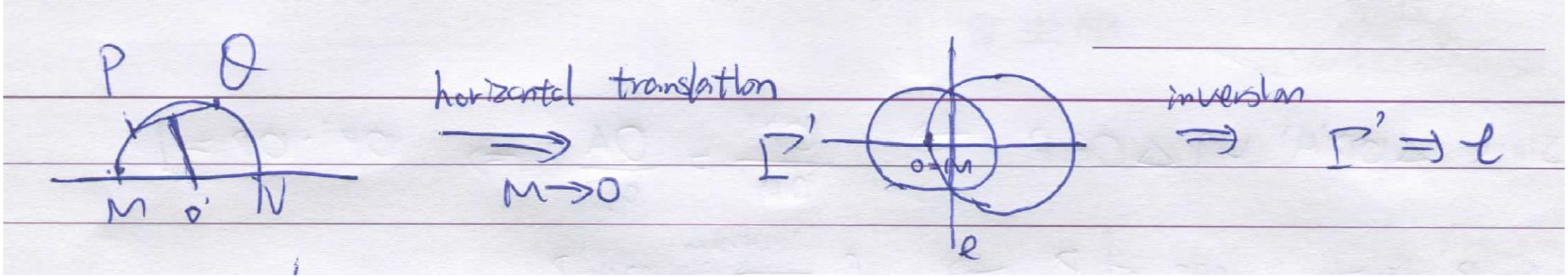
Lemma 2.2 Inversion Preserve Angles

- We will just consider the case of an angle α created by the intersection of a line l not intersecting the unit circle, and a line l’ through O.



Lemma 2.3 (Lines in Poincare Half-plane Model)

- Lines in the Poincare upper half plane model are (Euclidean) lines and (Euclidean) half circles that are perpendicular to the x-axis.



2.2Fractional Linear Transformations

$$T(z) = \frac{az+b}{cz+d}$$

$$T(-d/c) = \lim_{z \to -\frac{d}{c}} \frac{az+b}{cz+d} = \infty, \quad \text{if } c \neq 0,$$

$$T(\infty) = \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} \quad \text{if } c \neq 0,$$

$$T(\infty) = \lim_{z \to \infty} \frac{az+b}{cz+d} = \infty \quad \text{if } c = 0.$$

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note: Kγ=γ

$$k\gamma = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Thm 2.1 The set of fractional linear transformations forms a group under composition (matrix-multiplication).

$$M_{2\times 2}(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R \right\}$$

.2(R) which identifies

$$\mathrm{GL}_2(R) = \{\gamma \in M_{2\times 2}(R) \mid \det(\gamma) \neq 0\}$$

$$\mathrm{SL}_2(R) = \{\gamma \in \mathrm{GL}_2(R) \mid \det(\gamma) = 1\}$$

- Horizontal translation $T_a(x, y) = (x + a, y)$ $\tau_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$

- Inversion in the unit circle followed by reflection through x = 0

$$\varphi(x, y) = \left(\frac{-x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \qquad \sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Thm 2.2 *The group* $\mathrm{SL}_2(\mathbb{R})$ *is generated by* σ *and the maps* τ_a *for* $a \in \mathbb{R}$.

Reason:

$$\sigma\tau_t\sigma\tau_s\sigma\tau_r = \begin{bmatrix} 0 & -1 \\ 1 & t \end{bmatrix} \begin{bmatrix} -1 & -r \\ s & rs-1 \end{bmatrix} = \begin{bmatrix} -s & 1-rs \\ st-1 & rst-r-t \end{bmatrix}$$

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$r = \frac{b-1}{a} \quad \text{and} \quad t = \frac{-1-c}{a}.$$

2.3Cross Ratio:

The *cross ratio* of a, b, c, and d is defined to be

$$(a, b; c, d) = \frac{\dfrac{a-c}{a-d}}{\dfrac{b-c}{b-d}}.$$

then we get a fractional linear transformation:

$$T(z) = (z, a; b, c) = \frac{\dfrac{z-b}{z-c}}{\dfrac{a-b}{a-c}}. \qquad T(a) = 1, \quad T(b) = 0, \quad \text{and} \quad T(c) = \infty.$$

Example:

- If T sends i to i, ∞ to 3, 0 to-1/3, what is T?

- Set(w,i; 3,-1/3)=(z,i; ∞, 0)

Then we get

$$w = \frac{1}{10}\begin{bmatrix}3 & -1\\1 & 3\end{bmatrix} \equiv \begin{bmatrix}3 & -1\\1 & 3\end{bmatrix} z$$

2.4Let's consider Translations(not necessarily the horizontal case)

- Let P = a + bi and Q = c + di.
- (w, c+di; c ,∞)=(z, a + bi ; a, ∞)
- here we send the infinity to infinity, a to c.

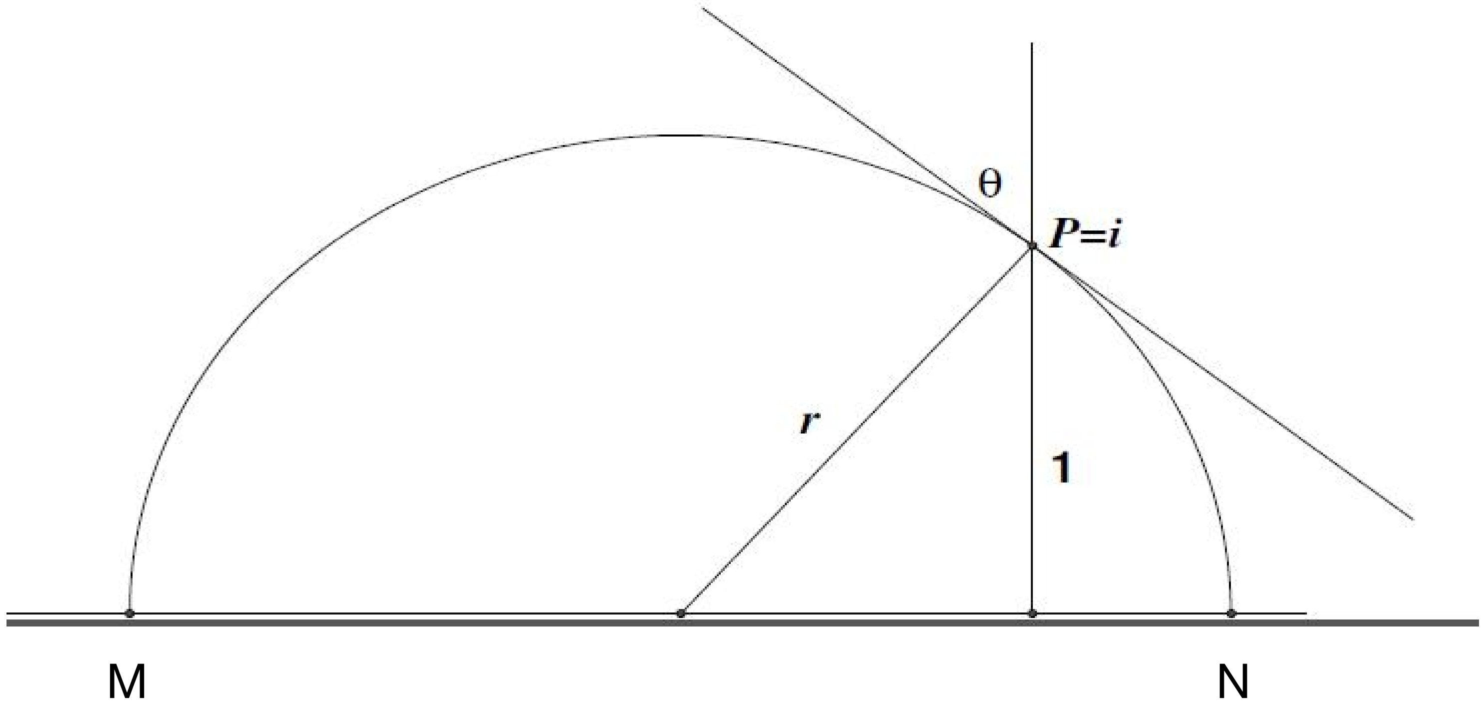
$$w = \frac{d(z-a)}{b} + c$$
$$= \begin{bmatrix} d & bc-ad \\ 0 & b \end{bmatrix} z.$$

- Does it have fixed point? No in Half-Plane Mc $z_0 = \frac{ad-bc}{d-b}$

2.5Rotation:

- Let this circle intersect the x-axis at points M and N
- Just need

$$(w, P; N, M) = (z, P; a, \infty).$$



$$w = \rho_\theta z = \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} z$$

Similar to Euclidean Case!

2.6Reflection

- We did see that the reflection through the imaginary axis is given by

$$R_0(x, y) = (-x, y),$$

t

$$R_0(z) = \mu\overline{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \overline{z}$$

- Now, to reflect through the line l in H , first use the appropriate isometry, γ1 to move the line l to the imaginary axis, then reflect and move the imaginary axis back to l:

$$\gamma_1^{-1}\mu\overline{\gamma_1 z} = \gamma_1^{-1}\mu\gamma_1\overline{z}.$$

- Note that μ^2 = 1 and detμ = -1

$$\gamma_1^{-1}\mu\gamma_1\overline{z} = \gamma_1^{-1}(\mu\gamma_1\mu)\mu\overline{z} = \gamma_2\mu\overline{z} = \gamma_2(-\overline{z})$$

- Hence every reflection can be written in the forn $\gamma(-\overline{z})$

for some $\gamma \in \mathrm{SL}_2(\mathbb{R})$

2.7Distance and length

- P and Q don't lie on a vertical line segment. We just rotate, then this is the transformation that sends P to i, M to 0 and N to ∞. Since the image of Q will lie on this line, Q is sent to some point 0+ci for some

$$(\sigma z, i; 0, \infty) = (z, P; M, N)$$

and in particular, since $\sigma(Q) = ci$ and $(\sigma z, i; 0, \infty) = \frac{\sigma z}{i}$, we get

$$c = (Q, P; M, N),$$

$$|PQ| = |\ln(Q, P; M, N)|.$$

2.8 Area of triangles

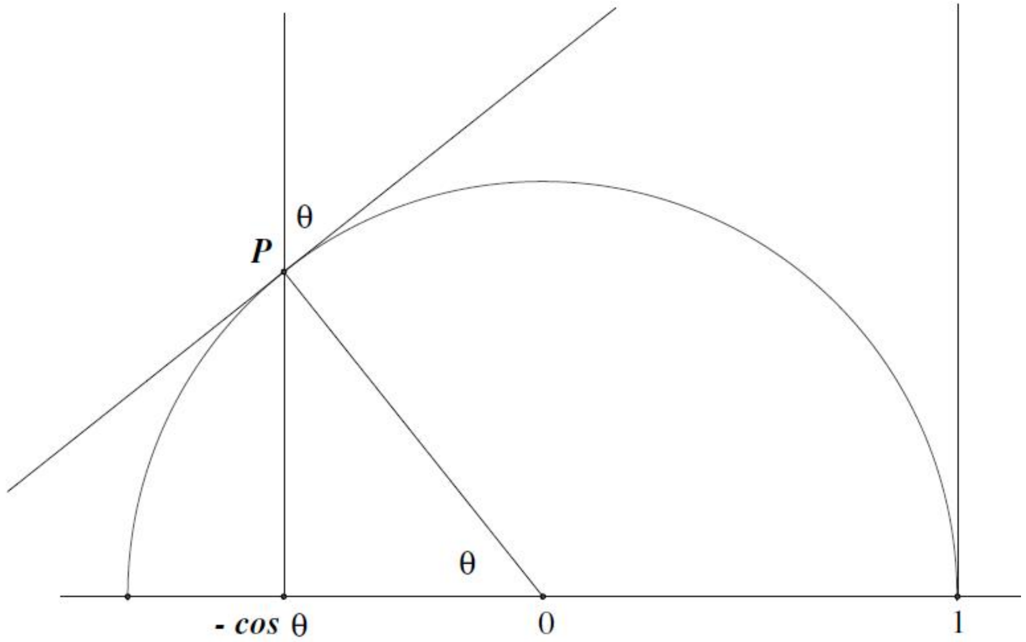
- Doubly asymptotic triangle: 2 vertices lie in infinity
- Triply asymptotic triangle: all 3 vertices lie in infinity

Lemma 2.4

- The area of a doubly asymptotic triar $P\Omega\Theta$ with po $\Omega\ and\ \Theta$ at infinity.

Then the area is

$$|\triangle P\Omega\Theta| = \pi - P,$$



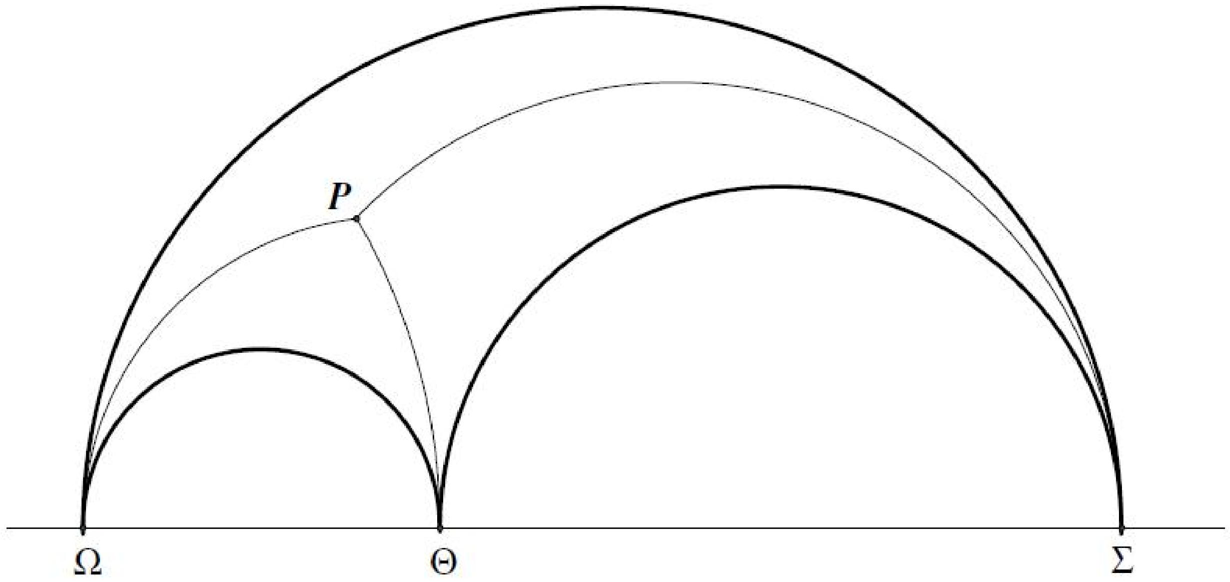
Since similar triangle has same area

$$\begin{aligned} A(\theta) &= \int_{-\cos\theta}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dx dy \\ &= \int_{-\cos\theta}^{1} \frac{dx}{\sqrt{1-x^2}} \\ &= \arccos(-x)\big|_{-\cos\theta}^{1} = \pi - \theta \end{aligned}$$

; the area element is given by $\dfrac{dxdy}{y^2}$.

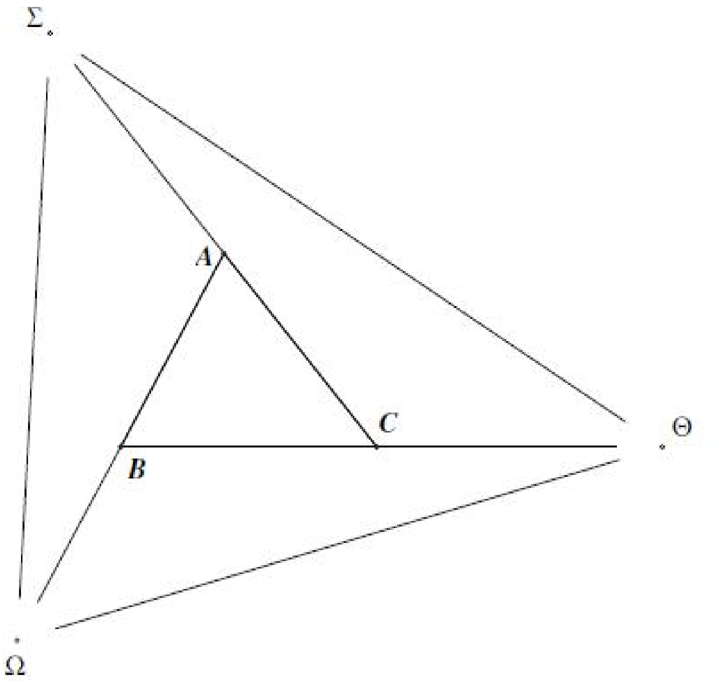
Corollary 2.1

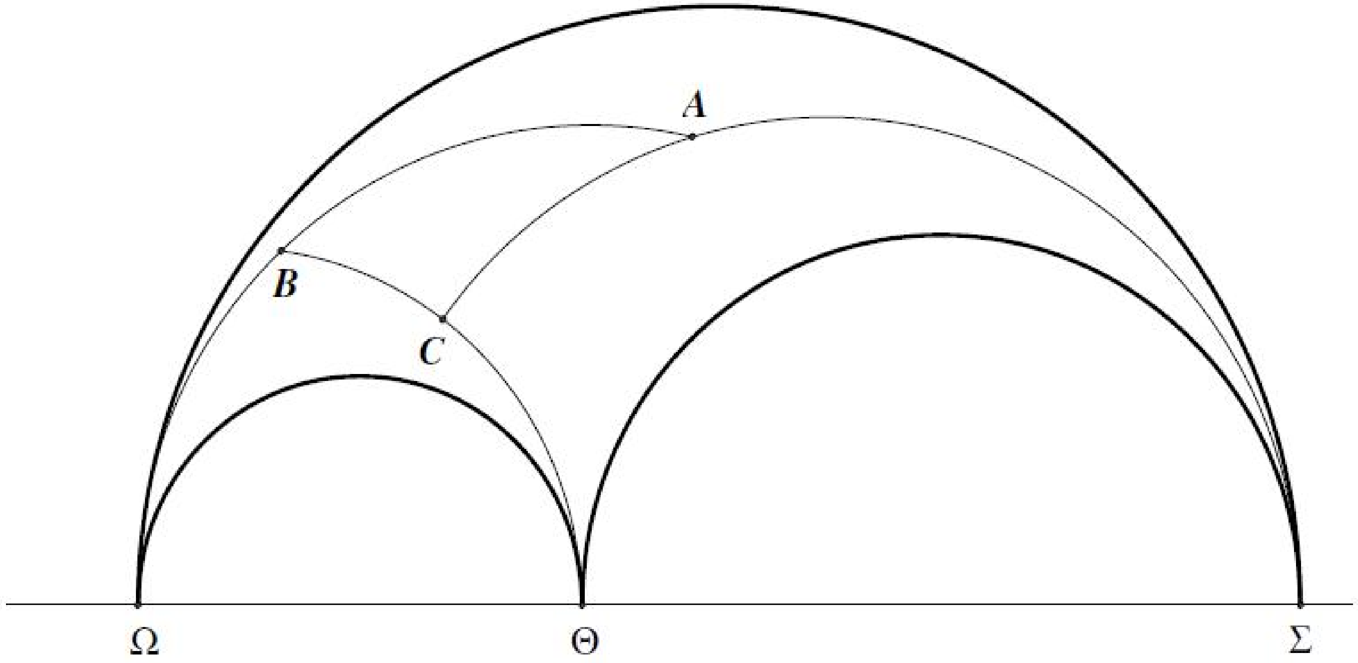
- Area of Triply asymptotic triangle is π.



Corollary 2.2

- Area of finite △ABC=π-(A+B+C)





Part 3 Poincare Disk Model

- Use the transform: $\phi = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

- This map sends 0 to -i, 1 to 1, and ∞ to i.

- Send upper half-plane to the interior of unit disk, denoted by D

RMK: Lines in D are circle arcs that are orthogonal to unit circle or diameters.

Lemma 3.1

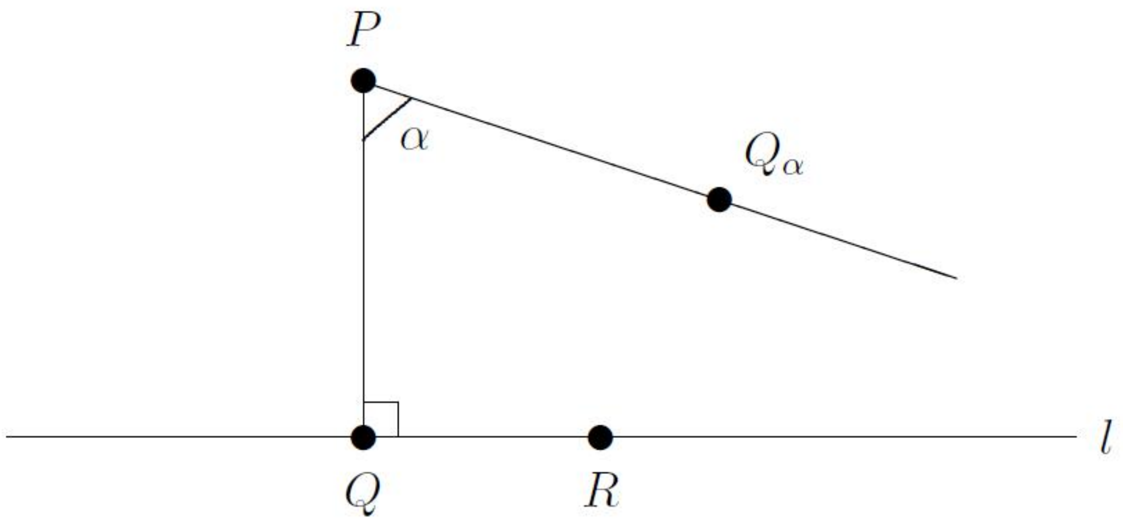
- If dp(O,B)=x, then d(O,B)=tanh(x/2)

If Ω and Λ are the ends of the diameter through OB then

$$
\begin{aligned}
x &= \log(O, B; \Omega, \Lambda) \\
e^x &= \frac{O\Omega \cdot B\Lambda}{O\Lambda \cdot B\Omega} \\
&= \frac{B\Lambda}{B\Omega} = \frac{1 + OB}{1 - OB}
\end{aligned}
$$

Angle of parallelism

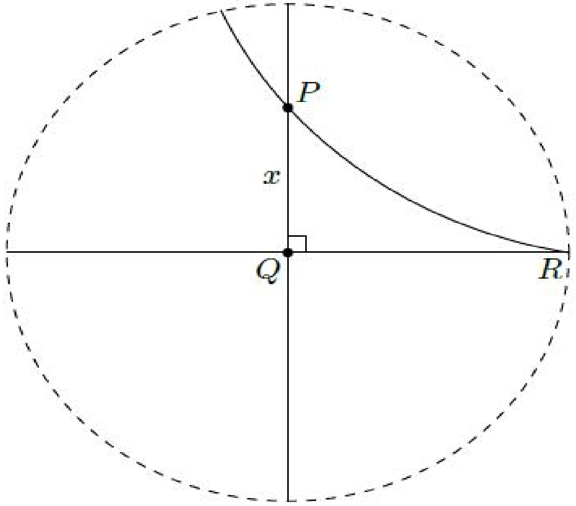
- the angle of parallelism ϕ, also known as Π(p), is the angle at one vertex of a right hyperbolic triangle that has two asymptotic parallel sides(intersect at infinity)



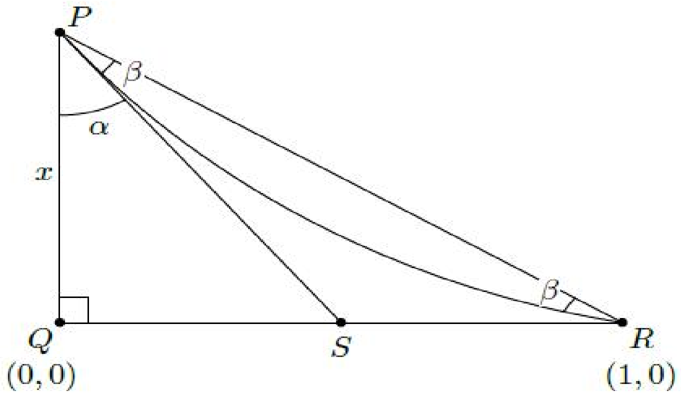
Thm 3.1 (Bolyai-Lobachevsky Theorem)

- In the Poincare model of hyperbolic geometry the angle of parallelism satisfies the equation

$$e^{-d} = \tan\left(\frac{\Pi(d)}{2}\right).$$



(a)



(b)

$$
\begin{aligned}
\pi &= \pi/2 + \alpha + 2\beta \\
\pi/4 - \beta &= \alpha/2.
\end{aligned}
$$

$$
\begin{aligned}
\tan(\alpha/2) &= \tan(\pi/4 - \beta) \\
&= \frac{1 - \tan(\beta)}{1 + \tan(\beta)} \\
&= \frac{1 - x}{1 + x}.
\end{aligned}
$$

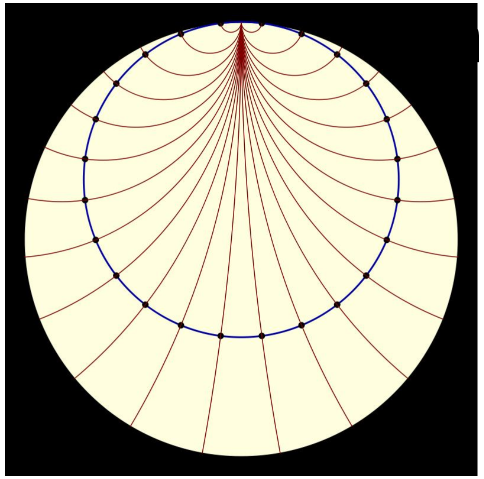
$$
e^{-d} = \frac{1 - x}{1 + x} = \tan\left(\frac{\alpha}{2}\right).
$$

Hypercycles and Horocycles

- Horocycle:

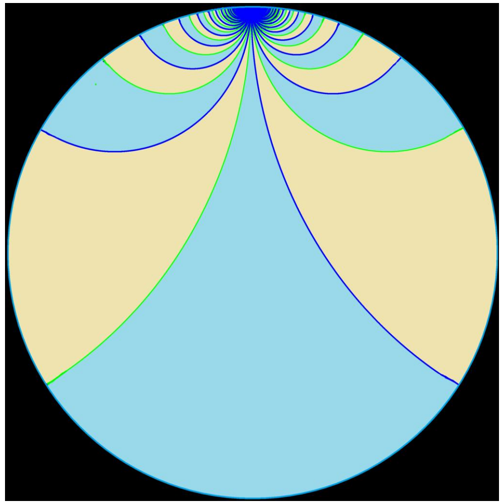
“Correspondence”: two lines intersect at infinity in the same direction Ω.

Then P and Q(on different lines) are correspondent if ∠PQΩ=∠QPΩ

The set consisting of  points Q is called a horocycle

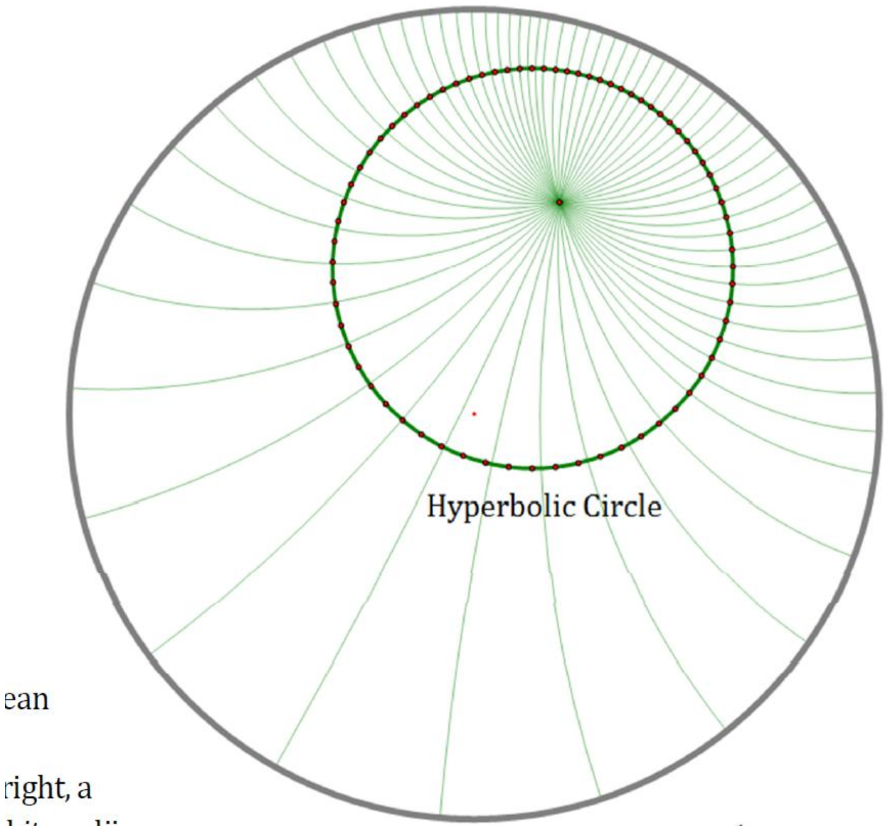
- Hypercycle:

Given a line l and a point P not on l, consider the set of all points Q on one side of l so that the perpendicular distance from Q to l is the same as the perpendicular distance from P to l.

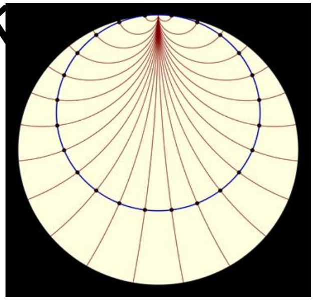


In Poincare Disk Model (4 types of cycles)

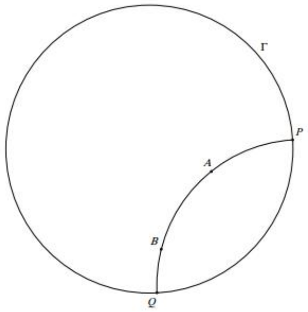
- Hyperbolic circle: a circle contained entirely in the unit disk



- Horocycle: circle tangent to unit disk



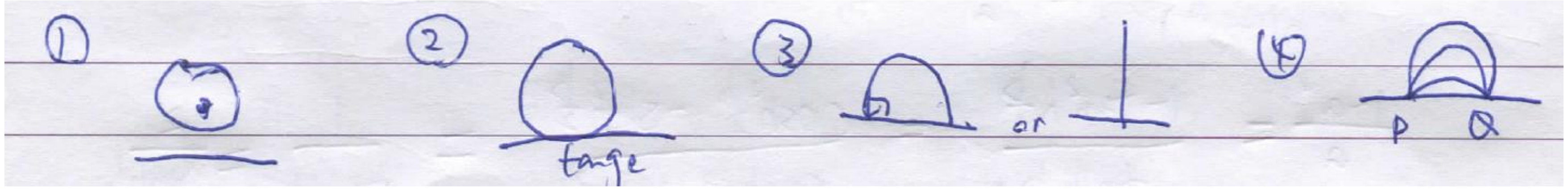
- Lines: circles orthogonal to unit disk or diameters



Otherwise Hypercycles

In upper half-plane model

- They are respectively circle, horocycle, line, hypercycle



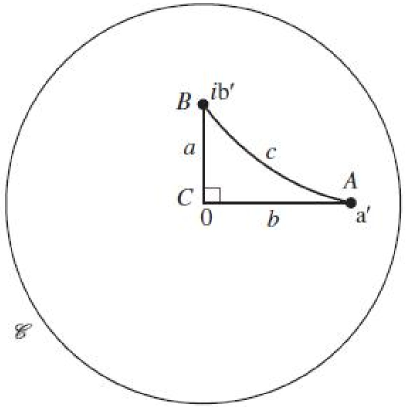
Part 4
4.1 some geometric laws

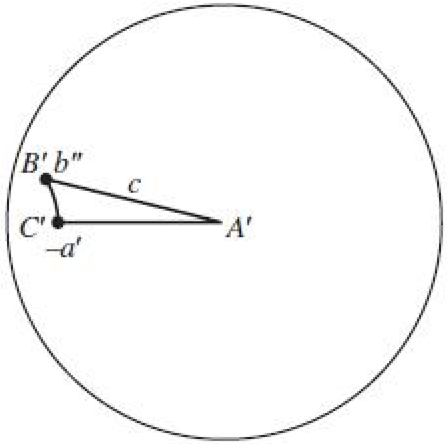
- Rt△ABC, ∠C=90 °
- Thm 4.1 **Pythagoras' Theorem**
  cosh(c)=cosh(a)*cosh(b)

  Thm 4.2 **Lobachevskii's Formula**
  tan(A)=tanh(a)/sinh(b)
  sin(A)=sinh(a)/sinh(c)
  cos(A)=tanh(b)/tanh(c)

Ideas:

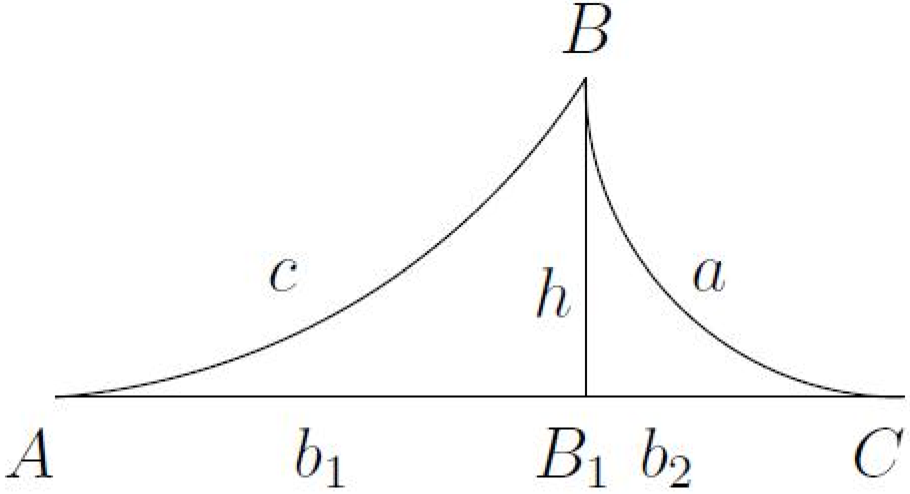
- Let C=0, B=ib’,A=a’
- Consider M(z)=(z-a’)/(1-a’z)
- Then we send A to O, B to b’’=(ib’-a’)/(1-a’ib’), C to-a’
- Then we use distance formula to calcula





Thm 4.3(Hyperbolic law of sines)

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}.$$



Hint:
By Thm 4.2
sinh(h) = sin(A) sinh(c)
sinh(h) = sin(C) sinh(a).

Thm 4.4(Hyperbolic law of cosines)

1st law:

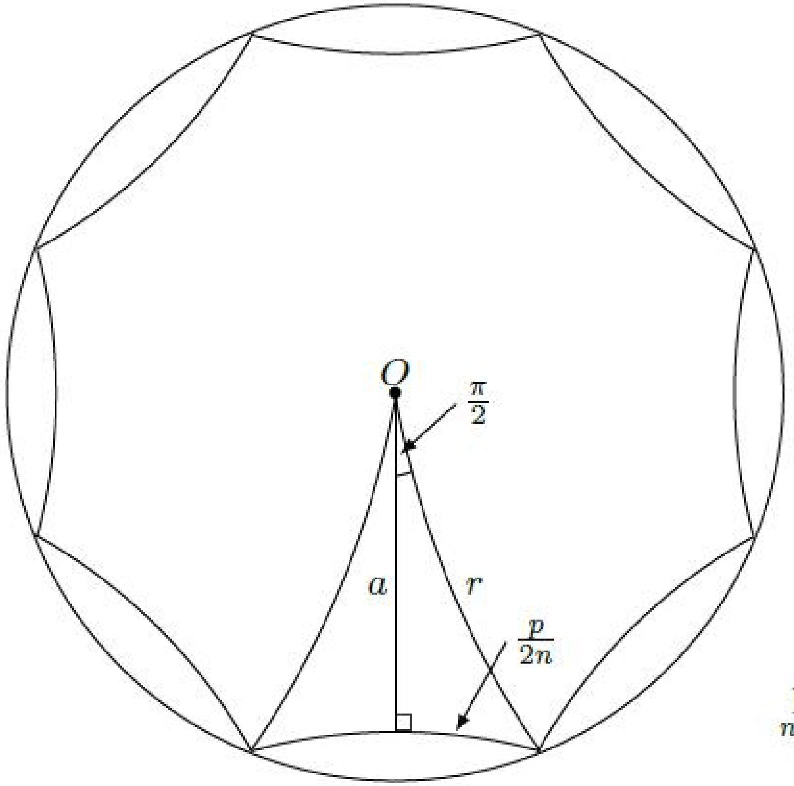
$\cosh(a) = \cosh(b) \cosh(c) - \sinh(b) \sinh(c) \cos(A)$.

2nd law:

$\cos(B) = -\cos(A) \cos(C) + \sin(A) \sin(C) \cosh(b)$.

4.2 Circumference and Area of a Circle

- Use similar method in Euclidean Geometry (use polygon)



By Thm 4.2

$$\sinh\left(\frac{p_n}{2n}\right) = \sinh r \cdot \sinh\left(\frac{\pi}{n}\right),$$

$$\frac{p_n}{2n}\left[1 + \frac{1}{3!}\left(\frac{p_n}{2n}\right)^2 + \frac{1}{5!}\left(\frac{p_n}{2n}\right)^4 + \cdots\right] = \frac{\pi}{n}\cdot\sinh r\left[1 - \frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right].$$

$$p_n\left[1 + \frac{1}{3!}\left(\frac{p_n}{2n}\right)^2 + \frac{1}{5!}\left(\frac{p_n}{2n}\right)^4 + \cdots\right] = 2\pi\cdot\sinh r\left[1 - \frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right].$$

$$\lim_{n\to\infty} p_n\left[1 + \frac{1}{3!}\left(\frac{p_n}{2n}\right)^2 + \frac{1}{5!}\left(\frac{p_n}{2n}\right)^4 + \cdots\right] = \lim_{n\to\infty} 2\pi\cdot\sinh r\left[1 - \frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right]$$

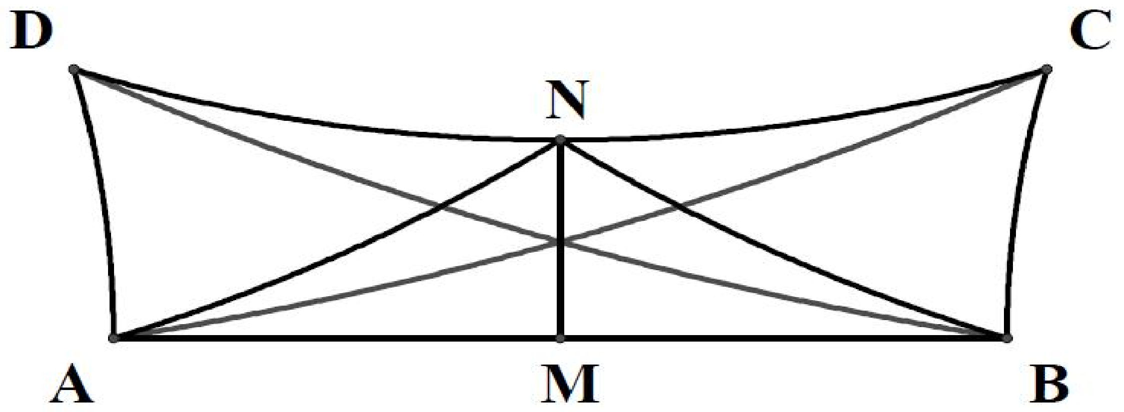
$$C = \lim_{n\to\infty} p_n = 2\pi\sinh r.$$

$C = 2\pi \sinh(r)$

- Similarly,
- Area= $4\pi \sinh^2(r/2)$

4.3 Saccheri Quadrilaterals& Lambert quadrilaterals

- Let S be a convex quadrilateral in which two adjacent angles are right angles.
- The segment joining these two vertices is called the base. The side opposite the base is the summit and the other two sides are called the sides. If the sides are congruent to one another then this is called a Saccheri quadrilateral. The angles containing the summi

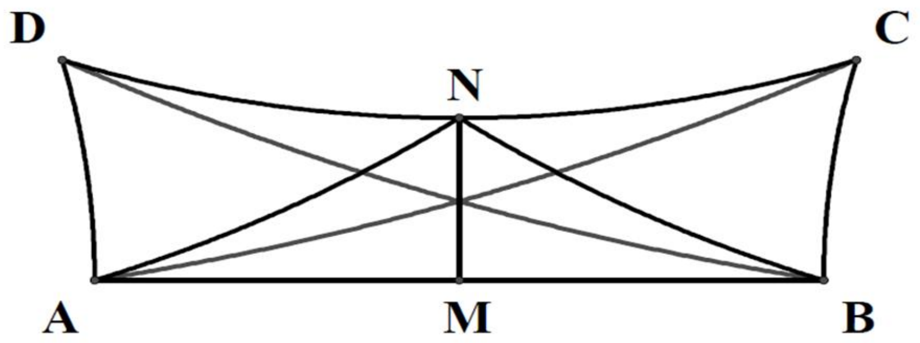


Properties:

i) the summit angles are congruent, and

ii) the line joining the midpoints of the base and the summit—called the **altitude**—*is perpendicular to both.*

Hint use triangle congruence to show

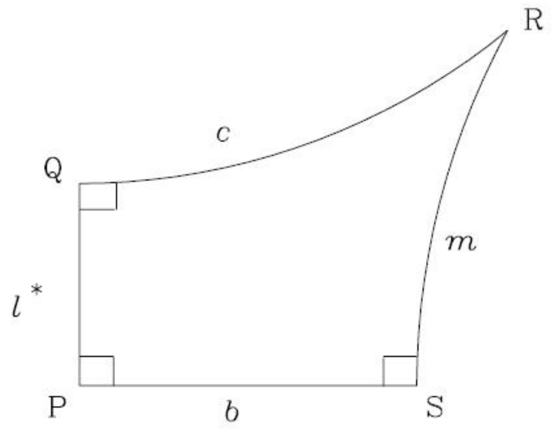


Lambert Quadrilaterals

- a convex quadrilateral three of whose angles are right angles is called a Lambert quadrilateral.

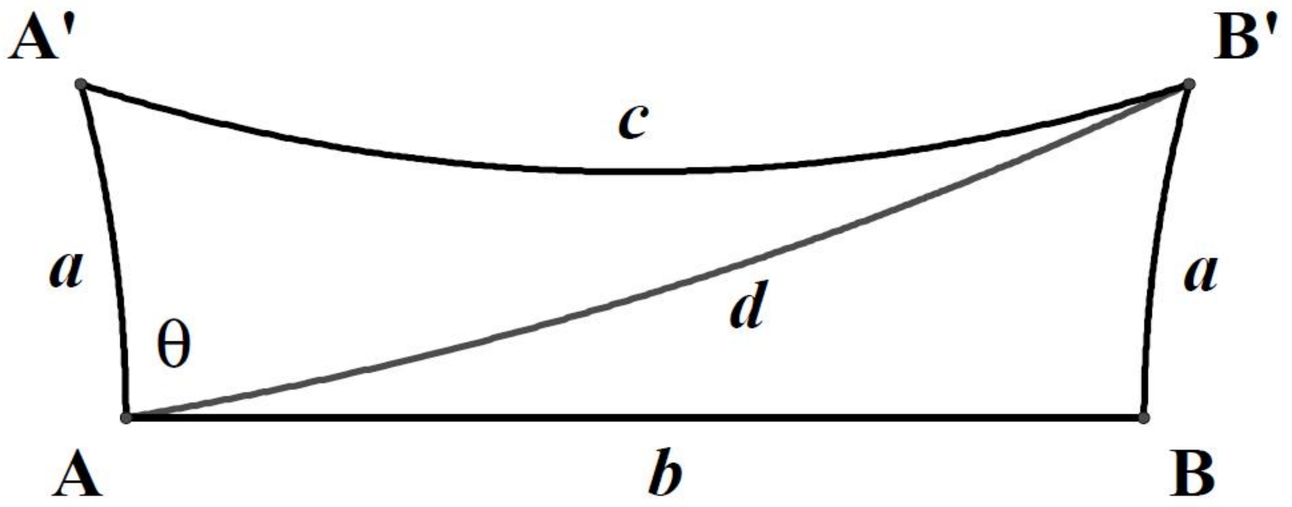
- Properties: The side adjacent to the acute angle of a Lambert quadrilateral is greater than its opposite side.

- Use exterior angle thm to prove



4.4 Constructing coordinate system

$$\sinh\frac{c}{2} = (\cosh a)\cdot(\sinh\frac{b}{2}).$$

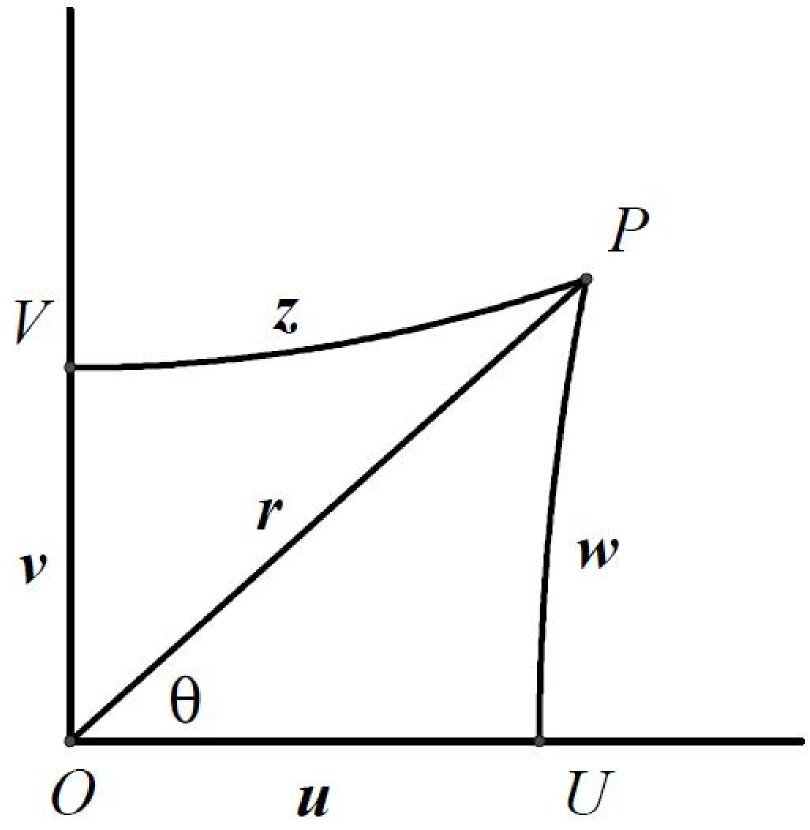


Use cos(θ)=sin(90-θ)=sinh(a)/sinh(d) and laws of cosine

- Then in Lambert quadrilateral, if c is the length of a side adjacent to the acute angle, a is the length of the other side adjacent to the acute angle, and b is the length of the opposite side, then

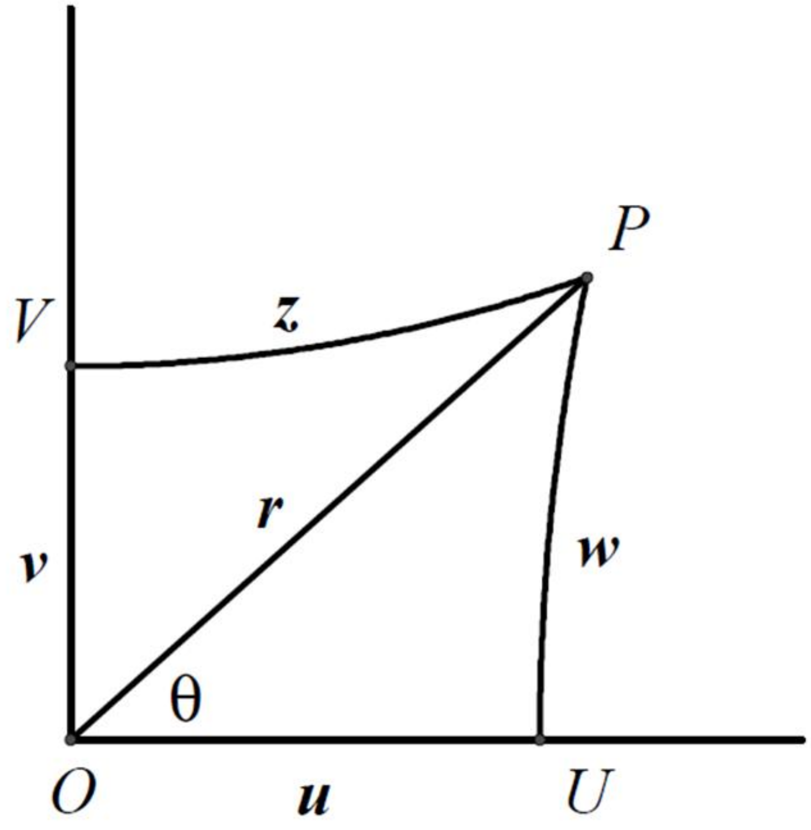
$$\sinh(c)= \cosh(a)*\sinh(b)$$

Then we could construct coordinate system



Sin(θ)=cos(90-θ)=tanh(v)/tanh(r);
cos(θ)=tanh(u)/tanh(r);

Then tanh^2(u)+tanh^2(v)=tanh^2(r)



We also set
x = tanh u; y = tanh v
T = cosh u coshw; X = xT; Y = yT;

- The ordered pair {OX;OY } is called a frame with axes OX and OY . With respect to this frame, we say the point P has
  - **axial coordinates** (u, v),
  - **polar coordinates** (r, θ),
  - **Lobachevsky coordinates** (u, w),
  - **Beltrami coordinates** (x, y),
  - **Weierstrass coordinates** (T, X, Y).

If a point has Beltrami coordinates (x, y) and $t = 1 + \sqrt{1 - x^2 - y^2}$, put

$$p = x/t \qquad q = y/t,$$

then (p, q) are the **Poincaré coordinates** of the point.

Thanks for watching!