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Energy stable semi-implicit schemes for the 2D Allen–Cahn and fractional Cahn–Hilliard equations

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In this work, we are interested in a class of numerical schemes for certain phase field models. It is well known that unconditional energy stability (energy decays in time regardless of the size of the time step) provides a fidelity check in practical numerical simulations. In recent work (Li, D. (2022b, Why large time-stepping methods for the Cahn–Hilliard equation is stable. *Math. Comp.*, 91, 2501–2515)), a type of semi-implicit scheme for the Cahn–Hilliard (CH) equation with regular potential was developed satisfying the energy-decay property. In this paper, we extend such semi-implicit schemes to the Allen–Cahn equation and the fractional CH equation with a rigorous proof of similar energy stability. Models in two spatial dimensions are discussed.

Keywords: Allen–Cahn; numerical analysis; phase field.

1. Introduction

1.1 *Introduction to the models and historical review*

In this work we consider two classic phase field models: Allen–Cahn (AC) and Cahn–Hilliard (CH) equations. The (AC) model was developed in Allen & Cahn (1972) by Allen and Cahn to study the competition of crystal grain orientations in an annealing process separation of different metals in a binary alloy; while the (CH) was introduced in Cahn & Hilliard (1958) by Cahn and Hilliard to describe the process of phase separation of different metals in a binary alloy. These equations are presented as follows:

$$\begin{cases} \partial_t u = \nu \Delta u - f(u), \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0, \end{cases} \tag{AC}$$

and

$$\begin{cases} \partial_t u = \Delta(-\nu \Delta u + f(u)), \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0, \end{cases} \tag{CH}$$

where $u(x,t)$ is a real valued function and values of u in $(-1,1)$ represent a mixture of the two phases, with -1 representing the pure state of one phase and $+1$ representing the pure state of the other phase. Vector position x is in the spatial domain Ω, which is oftentimes taken to be two or three dimensional periodic domain and t is the time variable. Here ν is a small parameter and we denote $\varepsilon = \sqrt{\nu}$ to represent

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an average distance over which phases mix. The energy term $f(u)$ is often chosen to be

$$f(u) = F'(u) = u^3 - u,\ F(u) = \frac{1}{4}(u^2-1)^2.$$

It is well known that, as $\varepsilon \to 0$, the limiting problem of (AC) is given by a mean curvature flow while the limiting problem of (CH) becomes Mullins–Sekerka problem; we refer to Ilmanen (1993) for AC and Pego (1989); Alikakos *et al.* (1994) for CH and a recent work for matrix-valued AC Fei *et al.* (2023). Both asymptotic and rigorous analysis are well-studied. For the other related models, the well-posedness of the fractional CH equation has been analyzed by Akagi *et al.* in Akagi *et al.* (2016) and Vázquez (2012, 2014) by Vázquez; the mass-conserving AC equation has been studied by Bronsard and Stoth in Bronsard & Stoth (1997); the time-fractional CH equation has been studied by Fritz *et al.* in Fritz *et al.* (2022); the logarithmic potential case have been studied by Li and Tang in Li & Tang (2021). Although the limiting behavior of these models are well known, there are related materials science models that are studied only numerically and this current work presents an idea about how to approach these models in an appropriate way numerically.

In this paper, we consider the spatial domain Ω to be the two dimensional 2π-periodic torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. In fact, our proof can be applied to more general settings such as Dirichlet and Neumann boundary conditions in a bounded domain. However, considering the periodic domain allows the use of efficient and accurate Fourier-spectral numerical methods; moreover, periodic domain is often appropriate for application questions, which involve the formation of micro-structure away from physical boundaries.

As is well known, the mass of the smooth solution of CH (CH) is conserved, i.e., $\frac{d}{\mathrm{d}t}M(t) \equiv 0$, $M(t) = \int_\Omega u(x,t)\,\mathrm{d}x$. This represents the conservation of the two phases in the mixture. In particular, $M(t) \equiv 0$ if $M(0) = 0$ and hence oftentimes zero-mean initial data (equal amounts of both phases) will be considered as a simpler, but representative case. The associated energy functional of (CH) is given by

$$E(u) = \int_\Omega \left(\frac{1}{2}\nu|\nabla u|^2 + F(u)\right)\,\mathrm{d}x. \tag{1.1}$$

The energy is often referred to as Ginzburg–Landau energy and in fact, the AC equation is also governed by the same energy, but AC is a L^2 gradient flow whereas CH is a H^{-1} gradient flow. More specifically, assume that $u(x,t)$ is a smooth solution with zero mean, one can deduce

$$\frac{d}{\mathrm{d}t}E(u(t)) + \int_\Omega |\nabla(-\nu\Delta u + f(u))|^2\,\mathrm{d}x = 0,$$

which implies the decay of the energy: $\frac{d}{\mathrm{d}t}E(u(t)) \le 0$. This thus provides an *a priori* H^1-norm bound and since the scaling-critical space for (CH) is L^2 in 2D and $\dot{H}^{\frac{1}{2}}$ in 3D, therefore (CH) admits a unique global solution following from standard global well-posedness theory. In this sense, the energy decay property is an important index for whether a numerical scheme is 'stable' or not. In comparison, AC equation does not share the mass conservation property; however, it still follows the energy decay property with the same energy functional. Moreover, the solution to the fractional CH equation (FCH) satisfies both mass conservation and energy properties; see Bosch & Stoll (2015) and Ainsworth & Mao (2017) for

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example. The fractional CH equation (FCH) is defined as the following:

$$\begin{cases} \partial_t u = \nu \Delta \left((-\Delta)^\alpha u + (-\Delta)^{\alpha-1} f(u) \right), \; 0 < \alpha \le 1 \\ u(x,0) = u_0. \end{cases} \tag{FCH}$$

The difficulty in dealing with this FCH model arises from the non-local behavior of the fractional Laplacian, where the fractional Laplacian on the torus is given from the Fourier side: for $x \in \mathbb{T}^d$, $(-\Delta)^\alpha f(x) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} |k|^{2\alpha} \hat{f}(k)\, e^{-ik\cdot x}$. The convention of Fourier series is given in the next section.

Various approaches have been developed to study numerical simulations on CH and related models. For example, in Gavish *et al.* (2012) and Christlieb *et al.* (2014), Christlieb *et al.* and Gavish *et al.* studied several gradient flow models using implicit time stepping methods, respectively; in He *et al.* (2007), He *et al.* studied large time stepping semi-discretized method for CH equation and proved the conditional energy stability; Xu and Tang studied large time stepping methods for epitaxial growth models in Xu & Tang (2006); Shen *et al.* studied energy stable schemes for several phase field models in Chen & Shen (1998) and Zhu *et al.* (1999); Bertozzi *et al.* studied biharmonic-modified forward time stepping methods for fourth order equation in Bertozzi *et al.* (2011); Eyre developed convex splitting method in Eyre (1998); Cai, Sun, Wang and Yang studied Cahn–Hilliard–Navier–Stokes System using finite element methods in Cai *et al.* (2023); Bai, Li and Wu developed low regular schemes in Bai *et al.* (2022); Bueno-Orovio, Kay and Burrage developed Fourier spectral methods for fractional reaction-diffusion equations in Bueno-Orovio *et al.* (2014).

Moreover, many other work studied related models numerically among which different approaches are applied to the time stepping including fully explicit (forward Euler scheme Bertozzi *et al.* (2011)), fully implicit (backward Euler) scheme Cheng *et al.* (2021a), semi-implicit (implicit-explicit) scheme Chen & Shen (1998); He *et al.* (2007); Li *et al.* (2022a); Liu *et al.* (2023), finite element scheme Feng & Prohl (2004); Li (2022a); Cai *et al.* (2023), convex splitting scheme Eyre (1998) and operator splitting schemes Li *et al.* (2022c,d). Different strategies are adopted for the spatial discretization including the Fourier-spectral method Chen & Shen (1998); He *et al.* (2007); Bueno-Orovio *et al.* (2014); Cheng *et al.* (2021b); Li *et al.* (2022b); Li (2022b); Wu & Yuan (2023). All the numerical approximations give accurate results to the values and qualitative features of the solution. One of the key features is the energy dissipation.

We hereby give a list of work in the historical review. From the analysis point of view, Feng and Prohl in Feng & Prohl (2004) introduced a semi-discrete in time and fully spatially discrete finite element method for CH equation(CH) where they obtained an error bound of size of powers of $1/\nu$. Explicit time-stepping schemes require strict time-step restrictions and do not obey energy decay in general. To guarantee the energy decay property and increase the time step, a good alternative is to use semi-implicit schemes in which the linear term is implicit (such as backward time differentiation) and the nonlinear term is treated explicitly. Having only a linear implicit at every time step has computational advantages, as suggested in Chen & Shen (1998), Chen and Shen considered a semi-implicit Fourier-spectral scheme for (CH). On the other hand, semi-implicit schemes can lose stability for large time steps and thus smaller time steps are needed in practice. To resolve this problem, semi-implicit methods with better stability have been introduced, e.g. Zhu *et al.* (1999); Xu & Tang (2006); He *et al.* (2007); Shen & Yang (2010a); Li *et al.* (2022a); Li (2022b). Specifically speaking, Zhu *et al.* (1999); Xu & Tang (2006); He *et al.* (2007) and Shen & Yang (2010b) give different semi-implicit Fourier-spectral schemes, which involve different stabilizing terms of different sizes, that preserve the energy decay property (we say these schemes are

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'energy stable'). However, those works either require a strong Lipschitz condition on the nonlinear source term, or require certain L^∞ bounds on the numerical solutions.

In the seminal works Li & Tang (2021); Li *et al.* (2022a); Li (2022b), Li *et al.* developed a large time-stepping semi-implicit Fourier-spectral scheme for CH equation and proved that it preserves energy decay with no *a priori* assumptions (unconditional stability). The proof uses tools from harmonic analysis in Bourgain & Li (2015a,b), and introduces a novel energy bootstrap scheme in order to obtain a L^∞-bound of the numerical solution. Their scheme for (CH) has the form:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Pi_N\Delta(f(u^n)), \ n \geq 0 \\ u^0 = \Pi_N u_0. \end{cases} \tag{1.2}$$

Here τ is the time step and A is a large coefficient for the $O(\tau)$ stabilizing term. As a result of their work, the energy decay can still be satisfied with a well-chosen large number A, with at least a size of $O(1/\nu|\log(\nu)|^2)$, or $c/\nu|\log(\nu)|^2$ for some positive constant c that depends on the initial conditions. This is not surprising, in fact Wu *et al.* obtained similar results in tumor growth model in Wu *et al.* (2014).

However, the methods in Li & Tang (2021); Li *et al.* (2022a) and Li (2022b) cannot be applied to the AC equation (AC) directly: this is due to the lack of mass conservation. The main contribution of our work is to extend their first-order semi-implicit scheme to the related AC equation (AC). Following the same path the fractional CH equation (FCH) can be studied as well. To be more specific, we consider the following stabilized semi-implicit scheme for (AC):

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - A(u^{n+1}-u^n) - \Pi_N f(u^n) \\ u^0 = \Pi_N u_0, \end{cases} \tag{1.3}$$

where τ is the time step and $A > 0$ is the coefficient for the $O(\tau)$ regularization term. For $N \geq 2$, we define

$$X_N = \operatorname{span}\left\{\cos(k\cdot x), \sin(k\cdot x): \ \ k=(k_1,k_2)\in\mathbb{Z}^2, \ |k|_\infty = \max\{|k_1|,|k_2|\} \leq N\right\}.$$

Define the L^2 projection operator $\Pi_N : L^2(\Omega) \to X_N$ by $(\Pi_N u - u, \phi) = 0 \ \ \forall\phi \in X_N$, where $(\cdot,\cdot)$ denotes the L^2 inner product on Ω. In other words, the projection operator Π_N is just the truncation of Fourier modes $|k|_\infty \leq N$. $\Pi_N u_0 \in X_N$ and by induction, we have $u^n \in X_N, \forall n \geq 0$. Similarly, the semi-implicit scheme for (FCH) is given by the following:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu(-\Delta)^{\alpha+1}u^{n+1} - (-\Delta)^\alpha A(u^{n+1}-u^n) - (-\Delta)^\alpha \Pi_N f(u^n) \\ u^0 = \Pi_N u_0. \end{cases} \tag{1.4}$$

We will show the numerical solutions in (1.3) and (1.4) are unconditionally energy stable and prove the L^2 error estimate.

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1.2 *Main results*

Our main results state below:

THEOREM 1.1 (Unconditional energy stability for AC). Consider (1.3) with $\nu > 0$ and assume $u_0 \in H^2(\mathbb{T}^2)$. Then there exists a constant β_0 depending only on the initial energy $E_0 = E(u_0)$ such that if

$$A \geq \beta \cdot \Big(\|u_0\|_{H^2}^2 + \nu^{-1}|\log \nu| + 1\Big), \ \beta \geq \beta_0$$

then $E(u^{n+1}) \leq E(u^n)$, $\forall n \geq 0$ and for any choice of the time step τ, where E is defined in (1.1).

REMARK 1.1. Note that here in Theorem 1.1 no mean zero assumption is needed for u_0 due to the lack of mass conservation. The choice of A is not optimal, in fact as suggested by the numerical experiments it suffices to choose $A = O(1)$ because in practice the time step τ is small (at least < 1) and thus the scheme gains better stability.

THEOREM 1.2. Let $\nu > 0$. Let $u_0 \in H^s$, $s \geq 4$ and $u(t)$ be the solution to the AC equation (AC) with initial data u_0. Let u^n be the numerical solution with initial data $\Pi_N u_0$ in (1.3). Assume A satisfies the same condition in Theorem 1.1. Define $t_m = m\tau$, $m \geq 1$. Then

$$\|u^m - u(t_m)\|_2 \leq A \cdot e^{C_1 t_m} \cdot C_2 \cdot \left(N^{-s} + \tau\right),$$

where $C_1 > 0$ depends only on (u_0, ν) and C_2 depends on (u_0, ν, s).

THEOREM 1.3 (Unconditional energy stability for FCH). Consider (1.4) with $\nu > 0$ and assume $u_0 \in H^2(\mathbb{T}^2)$ and obeys the zero-mean condition. Then there exists a constant β_0 depending only on the initial energy $E_0 = E(u_0)$ such that if

$$A \geq \beta \cdot \Big(\|u_0\|_{H^2}^2 + \nu^{-1}|\log \nu| + 1\Big), \ \beta \geq \beta_0$$

then $E(u^{n+1}) \leq E(u^n)$, $\forall n \geq 0$ and for any time step τ. Here E is defined above in (1.1).

REMARK 1.2. Here in Theorem 1.3, we require a zero-mean assumption on u_0 which implies u^n has mean zero for each n. This assumption will guarantee that negative fractional Laplacian is well defined. Here we use the notation $|\nabla|^{-\alpha} = (-\Delta)^{-\frac{\alpha}{2}}$ to denote the fractional Laplacian.

REMARK 1.3. It is worth mentioning that the stability results above in Theorem 1.1 and Theorem 1.3 are valid for any time step τ. Our choice of A is independent of τ as long as it has size of $O\left(1/\nu|\log(\nu)|\right)$. Note that the choice of A may not be optimal and further work can be done in this direction.

THEOREM 1.4. Let $\nu > 0$. Let $u_0 \in H^s$, $s \geq 4+4\alpha$ and $u(t)$ be the solution to the fractional CH equation (FCH) with initial data u_0. Let u^n be the numerical solution with initial data $\Pi_N u_0$ in (1.4). Assume A satisfies the same condition in Theorem 1.3. Define $t_m = m\tau$, $m \geq 1$. Then

$$\|u^m - u(t_m)\|_2 \leq A \cdot e^{C_1 t_m} \cdot C_2 \cdot \left(N^{-s} + \tau\right),$$

where $C_1 > 0$ depends only on (u_0, ν, α) and C_2 depends on (u_0, ν, α, s).

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The proof of Theorem 1.4 is very similar to the proof of Theorem 1.2, therefore we leave it to the readers.

REMARK 1.4. As a remark, in the fractional CH case, as $\alpha \to 0$, (FCH) becomes the zero-mass projected AC equation and for $\alpha = 1$, it coincides the original CH equation. Roughly speaking, the fractional CH equation is an interpolation of the zero-mass projected AC and CH equations. Here the zero-mass projected AC equation is defined as follows:

$$\begin{cases} \partial_t u = \Pi_0 \left(\nu \Delta u - f(u) \right) \\ u(x,0) = u_0, \end{cases} \tag{1.5}$$

where Π_0 is the zero mass projector, i.e., $\Pi_0(g) = g - \int_\Omega g \, \mathrm{d}x$, or $= \frac{1}{(2\pi)^d} \sum_{|k| \geq 1} \hat{g}(k)\, e^{ik\cdot x}$ from the Fourier side. The difference between (AC) and the zero-mass projected AC equation (1.5) results from the loss of mass conservation.

REMARK 1.5. More general cases can be discussed. To be more specific by defining a general 'gradient' operator $\mathcal{G}$, we can rewrite the equation as follows:

$$\begin{cases} \partial_t u = \mathcal{G} \left(\nu \Delta u - f(u) \right) \\ u(x,0) = u_0. \end{cases} \tag{1.6}$$

When $\mathcal{G} = id$, the identity map, (1.6) becomes the AC equation; when $\mathcal{G} = (-\Delta)^\alpha$, (1.6) becomes the fractional CH equation as discussed above. And the corresponding semi-implicit scheme is

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = \mathcal{G}\left(\nu \Delta u^{n+1} - f(u^n) \right) - A\mathcal{G}(u^{n+1} - u^n), \ n \geq 0 \\ u^0 = u_0. \end{cases} \tag{1.7}$$

The main result of this paper states that for any fixed time step τ, we can always define a large constant A independent of τ in (1.7), such that the numerical solution will be stable in the sense of satisfying the energy-decay condition for 'gradient' cases of AC and fractional CH in 2D. In fact, our method holds for more general cases including AC on 3D and higher order schemes; we postpone the discussion to a subsequent work.

1.3 *Organization of the presenting paper*

The presenting paper is organized as follows. In Section 2, we list the notation and preliminaries including several useful lemmas. The energy stability of the semi-implicit scheme of the 2D AC will be shown in Section 3 while the error estimate is given in Section 4. The fractional CH case will be discussed in Section 5.

2. Notation and preliminaries

Throughout this paper, for any two (non-negative in particular) quantities X and Y, we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. Similarly, $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually

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clear from the context and we will often suppress this dependence. We shall denote $X \lesssim_{Z_1,Z_2,\cdots,Z_k} Y$ if $X \le CY$ and the constant C depends on the quantities $Z_1, \cdots, Z_k$.

For any two quantities X and Y, we shall denote $X \ll Y$ if $X \le cY$ for some sufficiently small constant c. The smallness of the constant c is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

For a real-valued function $u : \Omega \to \mathbb{R}$ we denote its usual Lebesgue L^p-norm by

$$\|u\|_p = \|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \operatorname{esssup}_{x\in\Omega} |u(x)|, & p = \infty. \end{cases} \tag{2.1}$$

Similarly, we use the weak derivative in the following sense: For $u,\ v \in L^1_{loc}(\Omega)$, (i.e they are locally integrable); $\forall \phi \in C_0^\infty(\Omega)$, i.e ϕ is infinitely differentiable (smooth) and compactly supported; and

$$\int_\Omega u(x)\, \partial^\alpha \phi(x)\, \mathrm{d}x = (-1)^{\alpha_1+\cdots+\alpha_n} \int_\Omega v(x)\, \phi(x)\, \mathrm{d}x,$$

then v is defined to be the weak partial derivative of u, denoted by $\partial^\alpha u$. Suppose $u \in L^p(\Omega)$ and all weak derivatives $\partial^\alpha u$ exist for $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$, such that $\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, then we denote $u \in W^{k,p}(\Omega)$ to be the standard Sobolev space. The corresponding norm of $W^{k,p}(\Omega)$ is:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha u|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

For $p = 2$ case, we use the convention $H^k(\Omega)$ to denote the space $W^{k,2}(\Omega)$. We often use $D^m u$ to denote any differential operator $D^\alpha u$ for any $|\alpha| = m$: D^2 denotes $\partial^2_{x_i x_j} u$ for $1 \le i,j \le d$, as an example.

In this paper we use the following convention for Fourier expansion on $\mathbb{T}^d$:

$$f(x) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} \hat{f}(k)\, e^{ik\cdot x},\ \hat{f}(k) = \int_\Omega f(x)\, e^{-ik\cdot x}\, \mathrm{d}x.$$

Taking advantage of the Fourier expansion, we use the well-known equivalent H^s-norm and $\dot{H}^s$-semi-norm of function f by

$$\|f\|_{H^s} = \frac{1}{(2\pi)^{d/2}} \left(\sum_{k\in\mathbb{Z}^d} (1+|k|^{2s}) |\hat{f}(k)|^2 \right)^{\frac{1}{2}},\ \|f\|_{\dot{H}^s} = \frac{1}{(2\pi)^{d/2}} \left(\sum_{k\in\mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}}.$$

LEMMA 2.1 (Sobolev inequality on $\mathbb{T}^d$). Let $0 < s < d$ and $f \in L^q(\mathbb{T}^d)$ for any $\frac{d}{d-s} < p < \infty$, then

$$\| \langle \nabla \rangle^{-s} f\|_{L^p(\mathbb{T}^d)} \lesssim_{s,p,d} \|f\|_{L^q(\mathbb{T}^d)},\ \text{where } \frac{1}{q} = \frac{1}{p} + \frac{s}{d},$$

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where $\langle\nabla\rangle^{-s}$ denotes $(1-\Delta)^{-\frac{s}{2}}$ and $A \lesssim_{s,p,d} B$ is defined as $A \le C_{s,p,d}\, B$ where $C_{s,p,d}$ is a constant dependent on s, p and d.

REMARK 2.1. Note that the this Sobolev inequality is a variety of the standard version. Note that on the Fourier side the symbol of $\langle\nabla\rangle^{-s}$ is given by $(1+|k|^2)^{-\frac{s}{2}}$. In particular, $\|f\|_{\infty(\mathbb{T}^d)} \lesssim \|f\|_{H^2(\mathbb{T}^d)}$, known as Morrey's inequality. We refer the readers to Evans (2022) for the proof.

LEMMA 2.2 (Discrete Grönwall's inequality). Let $\tau > 0$ and $y_n \ge 0$, $\alpha_n \ge 0$, $\beta_n \ge 0$ for $n = 1, 2, 3 \cdots$. Suppose

$$\frac{y_{n+1}-y_n}{\tau} \le \alpha_n y_n + \beta_n, \ \forall\, n \ge 0.$$

Then for any $m \ge 1$, we have

$$y_m \le \exp\left(\tau \sum_{n=0}^{m-1} \alpha_n\right)\left(y_0 + \tau \sum_{k=0}^{m-1} \beta_k\right).$$

The proof of this lemma is standard; cf. Clark (1987).

3. Stability of a first-order semi-implicit scheme on the 2D AC equation

Recall that the AC equation (AC) is formulated as follows:

$$\begin{cases} \partial_t u = \nu \Delta u - f(u) \\ u(x,0) = u_0. \end{cases}$$

Here $f(u) = u^3 - u$, a regular potential, and the spatial domain Ω is taken to be the two dimensional $2\pi-$periodic torus $\mathbb{T}^2$. The corresponding energy is defined by $E(u) = \int_\Omega (\frac{\nu}{2}|\nabla u|^2 + F(u))\, \mathrm{d}x$, where $F(u) = \frac{1}{4}(u^2-1)^2$, the anti-derivative of $f(u)$. As is well known, the energy satisfies $E(u(t)) \le E(u(s)), \forall\, t \ge s$, which gives an *a priori* bound. Recall that we consider the stabilized semi-implicit scheme (1.3):

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu \Delta u^{n+1} - A(u^{n+1}-u^n) - \Pi_N f(u^n) \\ u^0 = \Pi_N u_0. \end{cases} \tag{3.1}$$

We aim to show Theorem 1.1. To start with we first introduce a log-type interpolation inequality:

LEMMA 3.1 (Log-type interpolation). For all $f \in H^s(\mathbb{T}^2)$, $s > 1$, then

$$\|f\|_\infty \le C_s \cdot \left(\|f\|_{\dot{H}^1}\sqrt{\log(\|f\|_{\dot{H}^s}+3)} + |\hat{f}(0)| + 1\right).$$

Here C_s is a constant that only depends on s.

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Proof. To prove the lemma, we write $f(x) = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}\hat{f}(k)\, e^{ik\cdot x}$, i.e., the Fourier series of f, which converge pointwisely to f. It then follows that

$$\begin{aligned}
\|f\|_\infty &\le \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}|\hat{f}(k)| \\
&\le \frac{1}{(2\pi)^2}\left(|\hat{f}(0)| + \sum_{0<|k|\le N}|\hat{f}(k)| + \sum_{|k|>N}|\hat{f}(k)|\right) \\
&\lesssim |\hat{f}(0)| + \sum_{0<|k|\le N}(|\hat{f}(k)||k|\cdot|k|^{-1}) + \sum_{|k|>N}(|\hat{f}(k)||k|^s\cdot|k|^{-s}) \\
&\lesssim |\hat{f}(0)| + \left(\sum_{0<|k|\le N}|\hat{f}(k)|^2|k|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{0<|k|\le N}|k|^{-2}\right)^{\frac{1}{2}} + \left(\sum_{|k|>N}|\hat{f}(k)|^2|k|^{2s}\right)^{\frac{1}{2}}\cdot\left(\sum_{|k|>N}|k|^{-2s}\right)^{\frac{1}{2}} \\
&\lesssim |\hat{f}(0)| + \frac{1}{N^{s-1}}\left(\sum_{|k|>N}|\hat{f}(k)|^2|k|^{2s}\right)^{\frac{1}{2}} + \left(\sum_{0<|k|\le N}|\hat{f}(k)|^2|k|^2\right)^{\frac{1}{2}}\cdot\sqrt{\log(N+3)} \\
&\lesssim |\hat{f}(0)| + \frac{1}{N^{s-1}}\|f\|_{\dot{H}^s} + \sqrt{\log(N+3)}\|f\|_{\dot{H}^1}.
\end{aligned}$$

If $\|f\|_{\dot{H}^s} \le 3$, we can simply take $N = 1$; otherwise take N^{s-1} close to $\|f\|_{\dot{H}^s}$. As a remark, this lemma can be viewed as a variation of the well-known log-type Bernstein's inequality; cf. Bahouri *et al.* (2011). □

We will prove Theorem 1.1 by induction. To start with, let us recall the numerical scheme (1.3):

$$\frac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - A(u^{n+1}-u^n) - \Pi_N f(u^n).$$

Here Π_N is truncation of Fourier modes of L^2 functions to $|k|_\infty \le N$. Multiply the equation by $(u^{n+1}-u^n)$ and integrate over Ω, one has

$$\frac{1}{\tau}\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 = \nu\int_{\mathbb{T}^2}\Delta u^{n+1}(u^{n+1}-u^n) - A\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 - \left(\Pi_N f(u^n), u^{n+1}-u^n\right).$$

Because u^n is periodic, (as $u^n \in X_N$), hence by integration by parts, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \nu\int_{\mathbb{T}^2}\nabla u^{n+1}\nabla(u^{n+1}-u^n) = -\left(\Pi_N f(u^n), u^{n+1}-u^n\right).$$

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Note $\nabla u^{n+1}\nabla(u^{n+1}-u^n) = \frac{1}{2}\left(|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2\right)$, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \frac{\nu}{2}\int_{\mathbb{T}^2}|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2 = -\Big(\Pi_N f(u^n), u^{n+1}-u^n\Big).$$

Moreover, every $u^n \in X_N$, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \frac{\nu}{2}\int_{\mathbb{T}^2}|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2 = -\Big(f(u^n), u^{n+1}-u^n\Big).$$

To proceed, by the fundamental theorem of calculus and integration by parts,

$$\begin{aligned} F(u^{n+1}) - F(u^n) &= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} f'(s)(u^{n+1}-s)\,\mathrm{d}s \\ &= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} (3s^2-1)(u^{n+1}-s)\,\mathrm{d}s \\ &= f(u^n)(u^{n+1}-u^n) + \frac{1}{4}(u^{n+1}-u^n)^2\Big(3(u^n)^2 + (u^{n+1})^2 + 2u^nu^{n+1} - 2\Big). \end{aligned}$$

Combine previous two equations, and denote $E(u^n)$ by E^n we have

$$\begin{aligned} &\left(\frac{1}{\tau}+A\right)\|u^{n+1}-u^n\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u^{n+1}\|_{L^2}^2 - \frac{\nu}{2}\|\nabla u^n\|_{L^2}^2 \\ &+ \int_{\mathbb{T}^2} F(u^{n+1}) - F(u^n) = \frac{1}{4}\Big((u^{n+1}-u^n)^2, 3(u^n)^2 + (u^{n+1})^2 + 2u^nu^{n+1} - 2\Big) \\ \text{Note } &\frac{\nu}{2}\|\nabla u^{n+1}\|_{L^2}^2 + \int_{\mathbb{T}^2} F(u^{n+1}) = E(u^{n+1}) = E^{n+1} \\ \Longrightarrow &\left(\frac{1}{\tau}+A+\frac{1}{2}\right)\|u^{n+1}-u^n\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + E^{n+1} - E^n \\ &= \frac{1}{4}\Big((u^{n+1}-u^n)^2, 3(u^n)^2 + (u^{n+1})^2 + 2u^nu^{n+1}\Big) \\ &\le \|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right). \end{aligned}$$

To show $E^{n+1} \le E^n$, clearly it suffices to show

$$\frac{1}{\tau} + A + \frac{1}{2} \ge \frac{3}{2}\max\Big\{\|u^n\|_\infty^2,\ \|u^{n+1}\|_\infty^2\Big\}. \tag{3.2}$$

Note that $E^0 = E(\Pi_N u_0)$ while $E_0 = E(u_0)$ and in general $E_0 \ne E^0$. Then the following statement holds.

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LEMMA 3.2. Suppose $E^0 = E(\Pi_N u_0)$ and $E_0 = E(u_0)$ as defined above, the following inequality holds:

$$\sup_N E(\Pi_N u_0) \lesssim 1 + E_0, \text{where } u_0 \in H^1(\mathbb{T}^2).$$

Proof. We rewrite $\Pi_N u_0$ as $\frac{1}{(2\pi)^2} \sum_{|k|\le N} \widehat{u_0}(k)\, e^{ik\cdot x}$, namely the Dirichlet partial sum of u_0.

$$\|\nabla \left(\Pi_N u_0\right)\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{(2\pi)^2} \sum_{|k|\le N} |k|^2 |\widehat{u_0}(k)|^2 \le \frac{1}{(2\pi)^2} \sum_{|k|\in\mathbb{Z}^2} |k|^2 |\widehat{u_0}(k)|^2 = \|\nabla \left(u_0\right)\|^2_{L^2(\mathbb{T}^2)}.$$

On the potential energy part, by the Sobolev inequality Lemma 2.1, $\|u_0\|_{L^4(\mathbb{T}^2)} \lesssim \|u_0\|_{H^1(\mathbb{T}^2)}$, this shows $u_0 \in L^4(\mathbb{T}^2)$ and hence the Dirichlet partial sum $\Pi_N u_0$ converges to u_0 in $L^4(\mathbb{T}^2)$. This leads to $\sup_N \|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} < \infty$. By the Uniform Boundedness Principle, we derive $\sup_N \|\Pi_N\| < \infty$, i.e., $\sup_N \|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} \le c\|u_0\|_{L^4(\mathbb{T}^2)}$ for an absolute constant c. Combining the two estimates above we prove the claim. It is also worth mentioning that the same claim holds for the 3D case with a similar proof. □

We rewrite the numerical scheme (1.3) as follows:

$$u^{n+1} = \frac{1 + A\tau}{1 + A\tau - \nu\tau\Delta} u^n - \frac{\tau}{1 + A\tau - \nu\tau\Delta} \Pi_N[f(u^n)]. \tag{3.3}$$

By the interpolation lemma (Lemma 3.1), to control $\|u^{n+1}\|_\infty$ and $\|u^n\|_\infty$, we may consider $\dot{H}^1$-norm and $\dot{H}^{\frac{3}{2}}$-norm together with 0th-mode $|\widehat{u^{n+1}}(0)|$. We start by estimating $|\widehat{u^{n+1}}(0)|$,

$$\begin{aligned}
|\widehat{u^{n+1}}(0)| &\le |\widehat{u^n}(0)| + \frac{\tau}{1 + A\tau} |\widehat{f(u^n)}(0)| \\
&\le |\widehat{u^n}(0)| + \frac{1}{A} |\widehat{f(u^n)}(0)| \\
&\le \left| \int_{\mathbb{T}^2} u^n \, dx \right| + \left| \int_{\mathbb{T}^2} u^n - (u^n)^3 \, dx \right| \\
&\lesssim 1 + \left| \int_{\mathbb{T}^2} (u^n)^2 \, dx \right|^{\frac{1}{2}} + \left| \int_{\mathbb{T}^2} (1 - (u^n)^2)^2 \, dx \right|^{\frac{1}{2}} \\
&\lesssim 1 + \sqrt{E^n}\,.
\end{aligned}$$

LEMMA 3.3. There is an absolute constant $c_1 > 0$ such that for any $n \ge 0$

$$\begin{cases}
\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \le c_1 \cdot \left(\dfrac{A+1}{\nu} + \dfrac{1}{\nu\tau} \right) \cdot (E^n + 1) \\
\|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} \le \left(1 + \dfrac{1}{A} + \dfrac{3}{A} \|u^n\|^2_\infty \right) \cdot \|u^n\|_{\dot{H}^1(\mathbb{T}^2)}.
\end{cases}$$

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Proof. As 0*th*-mode will not contribute to $\dot{H}^1$ norm and $\dot{H}^{\frac{3}{2}}$ norm, we can just consider Fourier modes $|k| \geq 1$ from the Fourier side.

Use the symbol $f \lesssim g$ to denote $f \leq c \cdot g$ with c being a constant. We then obtain that

$$\begin{cases} \dfrac{(1+A\tau)|k|^{\frac{3}{2}}}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{1+A\tau}{\nu\tau} \\ \dfrac{\tau|k|^{\frac{3}{2}}}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{\tau}{\tau\nu}|k|^{-\frac{1}{2}} = \dfrac{1}{\nu}|k|^{-\frac{1}{2}}. \end{cases}$$

Hence

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1+A\tau}{\nu\tau}\right)\|u^n\|_{L^2(\mathbb{T}^2)} + \frac{1}{\nu}\|\langle\nabla\rangle^{-\frac{1}{2}} f(u^n)\|_{L^2(\mathbb{T}^2)}. \tag{3.4}$$

Here the notaion $\langle\nabla\rangle^s = (1-\Delta)^{\frac{s}{2}}$, corresponds to the Fourier side $(1+|k|^2)^{s/2}$. Note that

$$\|u^n\|_{L^2(\mathbb{T}^2)} \lesssim \int_{\mathbb{T}^2} \frac{1}{4}(u^4 - 2u^2 + 1)\,\mathrm{d}x + 1 \lesssim E^n + 1$$

by Cauchy–Schwarz inequality. On the other hand, by the Sobolev inequality,

$$\begin{aligned} \|\langle\nabla\rangle^{-\frac{1}{2}} f(u^n)\|_{L^2(\mathbb{T}^2)} &\lesssim \|f(u^n)\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} = \|(u^n)^3 - u^n\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} \\ &= \left(\int_{\mathbb{T}^2} ((u^n)^3 - u^n)^{\frac{4}{3}}\,\mathrm{d}x\right)^{\frac{3}{4}} \\ &\lesssim E^n + 1. \end{aligned}$$

Therefore, (3.4) becomes

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1+A\tau}{\nu\tau} + \frac{1}{\nu}\right)(E^n + 1).$$

Similarly, we get

$$\begin{cases} \dfrac{(1+A\tau)|k|}{1+A\tau+\nu\tau|k|^2} \lesssim |k| \\ \dfrac{\tau|k|}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{\tau}{\tau A}|k| = \dfrac{1}{A}|k|. \end{cases}$$

This implies

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|f(u^n)\|_{\dot{H}^1(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|\nabla(f(u^n))\|_{L^2(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|(3(u^n)^2 - 1)\cdot(\nabla u^n)\|_{L^2(\mathbb{T}^2)} \end{aligned}$$

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$$\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \left(\frac{1}{A} + \frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1(\mathbb{T}^2)}$$
$$\lesssim \left(1 + \frac{1}{A} + \frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1(\mathbb{T}^2)}.$$

□

Proof of Theorem 1.1. Now we will complete the proof for Theorem 1.1 by induction:

Step 1: the induction $n \to n+1$ step. Assuming $E^n \leq E^{n-1} \leq \cdots \leq E^0$ and $E^n \leq \sup_N E(\Pi_N u_0)$, we will show $E^{n+1} \leq E^n$. This implies $\|u^n\|_{\dot{H}^1}^2 = \|\nabla u^n\|_{L^2}^2 \leq \frac{2E^n}{\nu} \leq \frac{2E^0}{\nu}$.

By Lemma 3.1, use the notation $f \lesssim_{E^0} g$ to denote that $f \leq C(E^0) \cdot g$ for some constant $C(E^0)$ depending only on E^0, we have

$$\|u^n\|_\infty^2 \lesssim \|u^n\|_{\dot{H}^1}^2 \left(\sqrt{\log\left(3 + c_1\left(\frac{1}{\nu\tau} + \frac{A+1}{\nu}\right)(E^n+1)\right)}\right)^2 + E^n + 1$$
$$\lesssim \frac{2E^0}{\nu}\left(1 + \log(A) + \log\left(\frac{1}{\nu}\right) + \left(\log\left(1 + \frac{1}{\tau}\right)\right)\right) + E^0 + 1$$
$$\lesssim_{E^0} \nu^{-1}\left(1 + \log(A) + \log\left(\frac{1}{\nu}\right)\right) + \nu^{-1}|\log(\tau)| + 1. \tag{3.5}$$

Define $m_0 := \nu^{-1}(1 + \log(A) + |\log(\nu)|)$, and note that $E^0 \leq \sup_N E(\Pi_N u_0) \lesssim E_0 + 1$, the inequality above (3.5) is then estimated as follows:

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0 + \nu^{-1}|\log(\tau)| + 1.$$

On the other hand by Lemma 3.3,

$$\|u^{n+1}\|_\infty \lesssim 1 + \|u^{n+1}\|_{\dot{H}^1}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)}$$
$$\lesssim 1 + \left(1 + \frac{1 + \|u^n\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)}$$
$$\lesssim_{E_0} 1 + \left(1 + \frac{m_0 + \nu^{-1}|\log(\tau)|}{A}\right)\left(\sqrt{\frac{1}{\nu}}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)}\right)$$
$$\lesssim_{E_0} 1 + \left(1 + \frac{m_0 + \nu^{-1}|\log(\tau)|}{A}\right)\left(\sqrt{m_0 + \nu^{-1}|\log(\tau)|}\right)$$
$$\lesssim_{E_0} 1 + \sqrt{m_0 + \nu^{-1}|\log(\tau)|} + \frac{\left(\sqrt{m_0 + \nu^{-1}|\log(\tau)|}\right)^3}{A}$$
$$\lesssim_{E_0} \sqrt{1 + \frac{m_0^3}{A^2} + m_0 + \nu^{-3}|\log(\tau)|^3}. \tag{3.6}$$

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The sufficient condition (3.2) thus becomes

$$\begin{cases} A + \dfrac{1}{2} + \dfrac{1}{\tau} \geq C(E_0)\left(m_0 + 1 + \dfrac{m_0^3}{A^2} + \nu^{-3}|\log(\tau)|^3\right) \\ m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right). \end{cases}$$

We now discuss two cases.

`Case 1:` $\frac{1}{\tau} \geq C(E_0)\nu^{-3}|\log(\tau)|^3$. In this case, we need to choose A such that

$$A \gg_{E_0} m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right),$$

where $B \gg_{E_0} D$ means there exists a large constant depending only on E_0. In fact, for $\nu \gtrsim 1$, we can take $A \gg_{E_0} 1$; if $0 < \nu \ll 1$, we will choose $A = C_{E_0} \cdot \nu^{-1}|\log \nu|$, where C_{E_0} is a large constant depending only on E_0. Therefore it suffices to choose

$$A = C_{E_0} \cdot \max\left\{\nu^{-1}|\log(\nu)|,\ 1\right\}. \tag{3.7}$$

`Case 2:` $\frac{1}{\tau} \leq C(E_0)\nu^{-3}|\log(\tau)|^3$. This implies $|\log(\tau)| \lesssim_{E_0} 1 + |\log(\nu)|$. Going back to equations (3.5), we have

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0,$$

as $\nu^{-1}|\log(\tau)|$ will be absorbed by m_0, where $m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right)$. Hence substituting this new bound into (3.6), we get

$$\begin{aligned} \|u^{n+1}\|_\infty &\lesssim 1 + \left(\frac{1 + \|u^n\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)} \\ &\lesssim_{E_0} 1 + \left(1 + \frac{m_0}{A}\right)\sqrt{\frac{1}{\nu}}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)} \\ &\lesssim_{E_0} 1 + \left(1 + \frac{m_0}{A}\right)\sqrt{m_0} \\ &\lesssim_{E_0} \sqrt{1 + \frac{m_0^3}{A^2} + m_0}. \end{aligned}$$

This shows it suffices to take

$$A \geq C_{E_0} m_0,$$

for a large enough constant C_{E_0} depending only on E_0. The same choice of A in `Case 1` (with a larger C_{E_0} if necessary) will still work.

Step 2: we check the induction base step $n = 1$. Clearly we only need to check

$$A + \frac{1}{2} + \frac{1}{\tau} \geq \|\Pi_N u_0\|_\infty^2 + \frac{1}{2}\|u^1\|_\infty^2.$$

By Lemma 3.3,

$$\begin{aligned}\|u^1\|_{\dot{H}^1} &\leq \left(1 + \frac{1}{A} + \frac{3}{A}\|\Pi_N u_0\|_\infty^2\right) \cdot \|u_0\|_{\dot{H}^1} \\ &\leq \left(1 + \frac{1}{A} + \frac{3}{A}\|\Pi_N u_0\|_\infty^2\right) \cdot \sqrt{\frac{2E^0}{\nu}}.\end{aligned}$$

As a result,

$$\begin{aligned}\|u^1\|_\infty &\lesssim 1 + |\widehat{u^1}(0)| + \|u^1\|_{\dot{H}^1}\sqrt{\log\left(3 + \|u^1\|_{\dot{H}^{\frac{3}{2}}}\right)} \\ &\lesssim 1 + \sqrt{E^0} + \left(1 + \frac{1}{A} + \frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\sqrt{\frac{2E^0}{\nu}}\sqrt{\log\left(3 + c_1\left(\frac{A+1}{\nu} + \frac{1}{\nu\tau}\right)(E_0 + 1)\right)} \\ &\lesssim_{E^0} 1 + \left(1 + \frac{1}{A} + \frac{3}{A}\|\Pi_N u_0\|_\infty^2\right) \cdot \nu^{-\frac{1}{2}} \cdot \sqrt{1 + \log(A) + |\log(\nu)| + |\log(\tau)|} \\ &\lesssim_{E_0} \left(1 + \frac{1}{A} + \frac{3}{A}\|\Pi_N u_0\|_\infty^2\right) \cdot \nu^{-\frac{1}{2}} \cdot \sqrt{1 + \log(A) + |\log(\nu)| + |\log(\tau)|}.\end{aligned}$$

Thus, we need to choose A such that

$$\begin{aligned}A + \frac{1}{2} + \frac{1}{\tau} \geq \|\Pi_N u_0\|_\infty^2 + C_{E_0} \cdot \left(1 + \frac{1}{A} + \frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)^2 \cdot \nu^{-1} \\ \cdot (1 + \log(A) + |\log(\nu)| + |\log(\tau)|),\end{aligned}$$

where C_{E_0} is a large constant depending only on E_0. Note that by Morrey's inequality,

$$\|\Pi_N u_0\|_{L^\infty(\mathbb{T}^2)} \lesssim \|\Pi_N u_0\|_{H^2(\mathbb{T}^2)} \lesssim \|u_0\|_{H^2(\mathbb{T}^2)}.$$

Then it suffices to take A such that

$$A \gg_{E_0} \|u_0\|_{H^2}^2 + \nu^{-1}|\log(\nu)| + 1. \tag{3.8}$$

This completes the induction and hence proves the theorem. □

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4. L^2 error estimate of the first-order semi-implicit scheme for the 2D AC equation

In this section, we will like to study the L^2 error between the semi-implicit numerical solution and the exact PDE solution to the AC equation in the domain $\mathbb{T}^2$ and eventually prove Theorem 1.2. To start with, we consider the auxiliary L^2 error estimate for near solutions.

4.1 *Auxiliary L^2 error estimate for near solutions*

Consider the following auxiliary system u^n and v^n for the first-order scheme:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - \Pi_N f(u^n) - A(u^{n+1}-u^n) + G_n^1 \\ \dfrac{v^{n+1}-v^n}{\tau} = \nu\Delta v^{n+1} - \Pi_N f(v^n) - A(v^{n+1}-v^n) + G_n^2 \\ u^0 = u_0,\ v^0 = v_0. \end{cases} \tag{4.1}$$

We define that $G_n = G_n^1 - G_n^2$.

Proposition 4.1. For solutions of (4.1), assume for some $N_1 > 0$,

$$\sup_{n\ge 0}(\|\nabla u^n\|_{L^2} + \|\nabla v^n\|_{L^2} + \|v^n\|_\infty) \le N_1.$$

Then for any $m \ge 1$,

$$\|u^m - v^m\|_2^2 \le \exp\left(m\tau\cdot\left(\frac{C(1+N_1^4)}{\nu}+\nu\right)\right) \cdot\left((1+A\tau)\,\|u_0-v_0\|_2^2 + \frac{\tau\nu}{2}\sum_{n=0}^{m-1}\|G_n\|_2^2\right), \tag{4.2}$$

where $C > 0$ is an absolute constant.

Proof. Write $e^n = u^n - v^n$. Then

$$\frac{e^{n+1}-e^n}{\tau} = \nu\Delta e^{n+1} - A(e^{n+1}-e^n) - \Pi_N\left(f(u^n) - f(v^n)\right) + G_n.$$

Taking L^2-inner product with e^{n+1} on both sides and recalling similar computations in previous section, we have

$$\frac{1}{2\tau}\left(\|e^{n+1}\|_{L^2}^2 - \|e^n\|_{L^2}^2 + \|e^{n+1}-e^n\|_{L^2}^2\right) + \nu\|\nabla e^{n+1}\|_{L^2}^2 + \frac{A}{2}(\|e^{n+1}\|_{L^2}^2 - \|e^n\|_{L^2}^2 + \|e^{n+1}-e^n\|_{L^2}^2) = \left(G_n,\ e^{n+1}\right) + \left(f(u^n) - f(v^n),\ \Pi_N e^{n+1}\right),$$

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where $(.,.)$ denotes the L^2 inner product and the last term is because Π_N is a self-adjoint operator $\big(\Pi_N f,\ g\big) = \big(f,\ \Pi_N g\big)$, since it is just an Nth Fourier mode truncation. By Hölder's inequality, we obtain that

$$\left|\left(G_n,\ e^{n+1}\right)\right| \leq \|e^{n+1}\|_{L^2}\|G_n\|_{L^2} \leq \frac{1}{2}\left(\nu\|G_n\|_{L^2}^2 + \frac{\|e^{n+1}\|_{L^2}^2}{\nu}\right).$$

Next, by the fundamental theorem of calculus, we have

$$\begin{aligned} f(u^n) - f(v^n) &= \int_0^1 f'(v^n + se^n)\,\mathrm{d}s\, e^n \\ &= (a_1 + a_2(v^n)^2)e^n + a_3 v^n (e^n)^2 + a_4 (e^n)^3, \end{aligned}$$

where a_i are constants can be computed. Note that we will denote C to be an absolute constant whose value may vary in different lines:

$$\begin{aligned} \left|\left((a_1 + a_2(v^n)^2)e^n,\ e^{n+1}\right)\right| &\leq C(1 + \|v^n\|_\infty^2)\|e^{n+1}\|_{L^2}\|e^n\|_{L^2} \\ &\leq \frac{\nu}{3}\|e^n\|_{L^2}^2 + \frac{C(1+N_1^4)}{\nu}\|e^{n+1}\|_{L^2}^2, \end{aligned}$$

by the Cauchy–Schwarz inequality. Moreover, the other two terms can be estimated similarly:

$$\begin{aligned} \left|\left(a_3 v^n (e^n)^2,\ e^{n+1}\right)\right| &\leq C\|v^n\|_\infty \|e^{n+1}\|_{L^2}\|e^n\|_{L^4}^2 \\ &\leq CN_1\|e^n\|_{L^2}\|\nabla e^n\|_{L^2}\|e^{n+1}\|_{L^2} \\ &\leq \frac{\nu}{3}\|e^n\|_{L^2}^2 + \frac{CN_1^4}{\nu}\|e^{n+1}\|_{L^2}^2, \end{aligned}$$

by the Sobolev embedding: $H^1(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$. Similarly, we have

$$\begin{aligned} \left|\left(a_4 (e^n)^3,\ e^{n+1}\right)\right| &\leq C\|e^{n+1}\|_{L^2}\|e^n\|_{L^6}^3 \\ &\leq C\|e^n\|_{L^2}\|\nabla e^n\|_{L^2}^2\|e^{n+1}\|_{L^2} \\ &\leq \frac{\nu}{3}\|e^n\|_{L^2}^2 + \frac{CN_1^4}{\nu}\|e^{n+1}\|_{L^2}^2. \end{aligned}$$

To simplify the formula, we sometimes use the notation $\|u\|_2$ to denote the L^2 norm. Collecting all estimates, we get

$$\frac{\|e^{n+1}\|_2^2 - \|e^n\|_2^2}{\tau} + A\left(\|e^{n+1}\|_2^2 - \|e^n\|_2^2\right) \leq \frac{\nu}{2}\|G_n\|_2^2 + \nu\|e^n\|_2^2 + \frac{C\left(1+N_1^4\right)}{\nu}\|e^{n+1}\|_2^2,$$

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where C is an absolute constant that can be computed exactly. Recalling that A is chosen larger than $O(\nu^{-1}|\log\nu|)$ for ν small, we derive that

$$\frac{\|e^{n+1}\|_2^2-\|e^n\|_2^2}{\tau}+\left(A-\frac{C(1+N_1^4)}{\nu}\right)\left(\|e^{n+1}\|_2^2-\|e^n\|_2^2\right)\le\frac{\nu}{2}\|G_n\|_2^2+\left(\frac{C(1+N_1^4)}{\nu}+\nu\right)\|e^n\|_2^2.$$

Define

$$\begin{aligned}
y_n &= \left(1+\left(A-\frac{C(1+N_1^4)}{\nu}\right)\tau\right)\|e^n\|_2^2,\\
\alpha &= \frac{C\left(1+N_1^4\right)}{\nu}+\nu,\\
\beta_n &= \frac{\nu}{2}\|G_n\|_2^2.
\end{aligned}$$

Then for ν small, we have

$$\frac{y_{n+1}-y_n}{\tau}\le\alpha y_n+\beta_n.$$

Applying discrete Grönwall's inequality in Lemma 2.2, we have

$$\begin{aligned}
\|u^m-v^m\|_2^2 =&\|e^m\|_2^2\le y_m\\
&\le\exp\left(m\tau\cdot\left(\frac{C(1+N_1^4)}{\nu}+\nu\right)\right)\\
&\quad\cdot\left(\left(1+\left(A-\frac{C(1+N_1^4)}{\nu}\right)\tau\right)\|u_0-v_0\|_2^2+\frac{\tau\nu}{2}\sum_{n=0}^{m-1}\|G_n\|_2^2\right)\\
&\le\exp\left(m\tau\cdot\left(\frac{C(1+N_1^4)}{\nu}+\nu\right)\right)\\
&\quad\cdot\left((1+A\tau)\,\|u_0-v_0\|_2^2+\frac{\tau\nu}{2}\sum_{n=0}^{m-1}\|G_n\|_2^2\right).
\end{aligned}\tag{4.3}$$

□

4.2 L^2 *error estimate of the 2D AC equation*

In this section, to simplify the notation, we will write $x\lesssim y$ if $x\le C(\nu,u_0)\,y$ for a constant C, depending on ν and u_0. We consider the system

$$\begin{cases}
\dfrac{u^{n+1}-u^n}{\tau}=\nu\Delta u^{n+1}-\Pi_N f(u^n)-A(u^{n+1}-u^n)\\
\partial_t u=\nu\Delta u-f(u)\\
u^0=\Pi_N u_0,\ u(0)=u_0.
\end{cases}\tag{4.4}$$

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In order to prove Theorem 1.2, it is clear that we shall estimate G_n introduced in (4.1) from the previous proposition. Note that for a one-variable function $h(t)$, one has the formula:

$$\begin{cases} \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h(t) = h(t_n) + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_{n+1}-t)\,\mathrm{d}t \\ \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h(t) = h(t_{n+1}) + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_n-t)\,\mathrm{d}t. \end{cases} \tag{4.5}$$

Using the formula (4.5) above and integrating the AC equation (AC) on the time interval $[t_n, t_{n+1}]$, we get

$$\frac{u(t_{n+1})-u(t_n)}{\tau} = \nu\Delta u(t_{n+1}) - A\left(u(t_{n+1})-u(t_n)\right) \\ - \Pi_N f(u(t_n)) - \Pi_{>N} f(u(t_n)) + G_n, \tag{4.6}$$

where $\Pi_{>N} = id - \Pi_N$, the large mode truncation operator, and

$$G_n = \frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t\Delta u\cdot(t_n-t)\,\mathrm{d}t - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t(f(u))(t_{n+1}-t)\,\mathrm{d}t + A\int_{t_n}^{t_{n+1}} \partial_t u\,\mathrm{d}t. \tag{4.7}$$

To bound $\|G_n\|_2$, we introduce some useful lemmas.

4.3 *Bounds on the AC exact solution and numerical solution*

LEMMA 4.1 (Maximum principle for smooth solutions to the AC equation). Let $T>0, d\le 3$ and assume $u\in C_x^2C_t^1(\mathbb{T}^d\times[0,T])$ is a classical solution to AC equation with initial data u_0. Then

$$\|u(.,\ t)\|_\infty \le \max\{\|u_0\|_\infty,\ 1\},\ \forall 0\le t\le T.$$

REMARK 4.1. As proved in Elliott & Zheng (1986), there exists a global $H_x^4C_t^1$ solution to AC equation. In fact, as pointed out by Li *et al.* in Li & Tang (2021); Li (2022b), the regularity will be better due to the smoothing effect. Therefore, we may assume a smooth solution and the proof can be found in the appendix.

LEMMA 4.2 (H^k boundedness of the exact solution). Assume $u(x,t)$ is a smooth solution to the AC equation in $\mathbb{T}^d$ with $d=1,2,3$ and the initial data $u_0\in H^k(\mathbb{T}^d)$ for $k\ge 2$. Then,

$$\sup_{t\ge 0}\|u(t)\|_{H^k(\mathbb{T}^d)} \lesssim_k 1, \tag{4.8}$$

where we omit the dependence on ν and u_0.

This lemma can be proved through the smoothing effect of the parabolic operator and we provide an alternative proof for the sake of completeness in the appendix.

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LEMMA 4.3 (Discrete version H^k boundedness). Suppose $u_0 \in H^k(\mathbb{T}^d)$ with $d \le 3$ and $k \ge 2$. Then, suppose u^n is the numerical solution that satisfies

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu \Delta u^{n+1} - A(u^{n+1}-u^n) - \Pi_N f(u^n) \\ u^0 = \Pi_N u_0, \end{cases}$$

then

$$\sup_{n\ge 0} \|u^n\|_{H^k(\mathbb{T}^d)} \lesssim_{A,k} 1.$$

REMARK 4.2. The bound on u^n is independent of time step τ and truncation number N.

REMARK 4.3. The H^k-boundedness for the exact solution and the numerical solution is similar in the sense that a smoothing effect will take place after a short time period. We postpone the proof in the appendix.

4.4 *Proof of L^2 error estimate of 2D AC equation*

Proof of Theorem 1.2. By Lemma 4.3, $\sup_{n\ge 0} \|u^n\|_\infty \lesssim 1$ using Morrey's inequality. Thus, the assumptions of Proposition 4.1 (auxiliary L^2 error estimate proposition) are satisfied. Recall that

$$G_n = \frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t \Delta u \cdot (t_n - t)\,\mathrm{d}t - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t(f(u))(t_{n+1}-t)\,\mathrm{d}t + A\int_{t_n}^{t_{n+1}} \partial_t u\,\mathrm{d}t.$$

Then we can estimate that

$$\begin{aligned} \|G_n\|_2 &\lesssim \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \|\partial_t(f(u))\|_2\,\mathrm{d}t + A\int_{t_n}^{t_{n+1}} \|\partial_t u\|_2\,\mathrm{d}t \\ &\lesssim \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2\,\mathrm{d}t \cdot \left(A + \|f'(u)\|_{L_t^\infty L_x^\infty}\right) \\ &:= I_1 + I_2. \end{aligned} \tag{4.9}$$

Note that $\partial_t u = \nu\Delta u - u + u^3$ and hence by Lemma 4.2,

$$\|\partial_t u\|_2 \lesssim 1, \quad \|f'(u)\|_\infty \lesssim 1.$$

Recall the energy decay property:

$$\frac{dE}{\mathrm{d}t} = -\|\partial_t u\|_2^2.$$

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This shows

$$\int_0^\infty \|\partial_t u\|_2^2 \,\mathrm{d}t \lesssim 1.$$

Also note that by Lemma 4.2, we have

$$\int_0^T \|\partial_t \Delta u\|_2^2 \,\mathrm{d}t \lesssim 1+T.$$

Therefore, we can estimate (4.9) as follows

$$I_1 = \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \,\mathrm{d}t \lesssim \left(\int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \cdot \sqrt{\tau}.$$

Similarly for I_2, we obtain that

$$I_2 \lesssim (1+A)\cdot \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \,\mathrm{d}t \lesssim (1+A)\cdot \left(\int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \cdot \sqrt{\tau}.$$

Hence for $t_m \geq 1$,

$$\begin{aligned}
\sum_{n=0}^{m-1} \|G_n\|_2^2 &\lesssim \sum_{n=0}^{m-1} \left((I_1)^2 + (I_2)^2\right)\\
&\lesssim \sum_{n=0}^{m-1} \left(\tau \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \,\mathrm{d}t + (1+A)^2 \tau \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \,\mathrm{d}t\right)\\
&\lesssim \tau \int_0^{t_m} \|\partial_t \Delta u\|_2^2 \,\mathrm{d}t + (1+A)^2 \tau \int_0^{t_m} \|\partial_t u\|_2^2 \,\mathrm{d}t\\
&\lesssim \tau(1+t_m) + (1+A)^2\tau\\
&\lesssim (1+A)^2 \tau \cdot (1+t_m). \qquad (4.10)
\end{aligned}$$

On the other hand, by the high Sobolev bound lemma (Lemma 4.2) $\sup_{t\geq 0} \|u(t)\|_{H^s} \lesssim_s 1$, we have $\sup_{n\geq 0} \|f(u(t_n))\|_{H^s} \lesssim_s 1$. We can then derive that

$$\begin{aligned}
\|\Pi_{>N} f(u(t_n))\|_2^2 &= \sum_{|k|>N} \left|\widehat{f(u(t_n))}(k)\right|^2\\
&\leq \sum_{|k|>N} |k|^{2s} \left|\widehat{f(u(t_n))}(k)\right|^2 \cdot |k|^{-2s}\\
&\lesssim N^{-2s} \cdot \sum_{|k|>N} |k|^{2s} \left|\widehat{f(u(t_n))}(k)\right|^2\\
&\lesssim N^{-2s} \cdot \|f(u(t_n))\|_{H^s}^2\\
&\lesssim N^{-2s},
\end{aligned}$$

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thus

$$\sum_{n=0}^{m-1} \|\Pi_{>N} f(u(t_n))\|_2^2 \lesssim_s m \cdot N^{-2s} \lesssim \frac{t_m N^{-2s}}{\tau}.$$

Therefore,

$$\tau \sum_{n=0}^{m-1} \left(\|G_n\|_2^2 + \|\Pi_{>N} f(u(t_n))\|_2^2 \right) \lesssim_s (1+t_m)(\tau^2 + N^{-2s})(1+A)^2.$$

Similarly, we have

$$\|u^0 - u(0)\|_2^2 = \|\Pi_N u_0 - u_0\|_2^2 \lesssim N^{-2s}.$$

Applying the auxiliary solutions estimate in Proposition 4.1 and noting that $t_m = m\tau$, we can get

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \left(N^{-2s} + \tau \cdot N^{-2s} + (1+t_m)(\tau^2 + N^{-2s}) \right).$$

Note that

$$\begin{cases} \tau \cdot N^{-2s} \lesssim \tau^2 + N^{-4s} \lesssim \tau^2 + N^{-2s} \\ 1 + t_m \lesssim e^{C' t_m}, \end{cases}$$

which leads to

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \left(N^{-2s} + \tau^2 \right).$$

Thus,

$$\|u^m - u(t_m)\|_2 \leq (1+A) \cdot C_2 \cdot e^{C_1 t_m} \left(N^{-s} + \tau \right), \tag{4.11}$$

where $C_1 > 0$ is a constant, depending on $\nu\, u_0$; $C_2 > 0$ is a constant, depending on s, ν and u_0. This completes the proof of L^2 error estimate. □

5. Stability of a first-order semi-implicit scheme for the 2D fractional CH equation

In this section, we will show Theorem 1.4. As mentioned earlier, the fractional CH equation behaves as an 'interpolation' between AC equation and original CH equation:

$$\begin{cases} \partial_t u = \nu \Delta \left((-\Delta)^{\alpha} u + (-\Delta)^{\alpha-1} f(u) \right),\ 0 < \alpha \leq 1 \\ u(x,0) = u_0. \end{cases}$$

In this section, we stick to the same region, two dimensional 2π-periodic torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. $f(u) = u^3 - u$ and the energy $E(u) = \int_{\mathbb{T}^2} \left(\frac{\nu}{2}|\nabla u|^2 + F(u) \right) \, \mathrm{d}x$, with $F(u) = \frac{1}{4}(u^2-1)^2$. Recall that the

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semi-implicit scheme (1.4) is given by the following:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu(-\Delta)^{\alpha+1}u^{n+1} - (-\Delta)^{\alpha}A(u^{n+1}-u^n) - (-\Delta)^{\alpha}\Pi_N f(u^n) \\ u^0 = \Pi_N u_0. \end{cases}$$

Proof of Theorem 1.4. The proof adopts a similar computation given in previous section. We recall the scheme (1.4):

$$\frac{u^{n+1}-u^n}{\tau} = -\nu(-\Delta)^{\alpha+1}u^{n+1} - (-\Delta)^{\alpha}A(u^{n+1}-u^n) - (-\Delta)^{\alpha}\Pi_N f(u^n).$$

Now we multiply the equation by $(-\Delta)^{-\alpha}(u^{n+1}-u^n)$ and apply the fundamental theorem of calculus as in section 3. We then obtain

$$\begin{aligned} &\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\left(\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2\right) \\ &+A\|u^{n+1}-u^n\|_{L^2}^2 = -\left(f(u^n), u^{n+1}-u^n\right). \end{aligned}$$

This then implies that

$$\begin{aligned} &\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \left(A+\frac{1}{2}\right)\|u^{n+1}-u^n\|_{L^2}^2 + E^{n+1} - E^n \\ &\le \|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right). \end{aligned} \tag{5.1}$$

It is clear that the first two norms $\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2$ and $\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2$ will be hard to control as we will expect more help from $\|u^{n+1}-u^n\|_{L^2}^2$. □

Lemma. 5.1. There exists a constant $C_{\alpha\nu\tau}$ that is determined by α, ν and τ, such that

$$\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2 \ge C_{\alpha\nu\tau}\|u^{n+1}-u^n\|_{L^2(\mathbb{T}^2)}^2.$$

Proof. It is natural to examine the above norms $\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2$ and $\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2$ on the Fourier side. Then we obtain that

$$\begin{aligned} &\frac{1}{\tau}\sum_{k\neq 0}|k|^{-2\alpha}|\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 + \frac{\nu}{2}\sum_{k\neq 0}|\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 \\ &= \sum_{k\neq 0}|\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 \cdot \left(\frac{|k|^{-2\alpha}}{\tau} + \frac{\nu|k|^2}{2}\right). \end{aligned}$$

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We apply standard Young's inequality for product to estimate: $ab \le \frac{a^\gamma}{\gamma} + \frac{b^\beta}{\beta}$, with $\frac{1}{\gamma} + \frac{1}{\beta} = 1$. We then take $a = |k|^p$, $b = |k|^q$, where $p + q = 0$. To fulfill the condition, we choose γ, β, p and q as follows:

$$\begin{cases} p = \dfrac{-2\alpha}{\alpha+1} \\ q = \dfrac{2\alpha}{\alpha+1} \\ \gamma = \alpha + 1 \\ \beta = \dfrac{\alpha+1}{\alpha}. \end{cases} \implies \begin{cases} -2\alpha = p\gamma \\ 2 = q\beta \end{cases}$$

Therefore, we have

$$\begin{cases} a^\gamma = |k|^{p\gamma} = |k|^{-2\alpha} \\ b^\beta = |k|^{q\beta} = |k|^2. \end{cases}$$

As a result, we obtain that

$$\begin{aligned} &\sum_{k\neq 0} |\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 \cdot \left(\frac{|k|^{-2\alpha}}{\tau} + \frac{\nu |k|^2}{2} \right) \\ &= \sum_{k\neq 0} |\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 \cdot \left[\frac{\alpha+1}{\tau} \cdot \left(\frac{|k|^{-2\alpha}}{\alpha+1} \right) + \frac{\nu(\alpha+1)}{2\alpha} \cdot \left(\frac{|k|^2}{\frac{\alpha+1}{\alpha}} \right) \right] \\ &\ge \sum_{k\neq 0} |\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 \cdot \left(\frac{\alpha+1}{\tau} \right)^{\frac{1}{\alpha+1}} \cdot \left(\frac{\nu(\alpha+1)}{2\alpha} \right)^{\frac{\alpha}{\alpha+1}}. \end{aligned}$$

Clearly it suffices to take $C_{\alpha\tau\nu} = \left(\frac{\alpha+1}{\tau}\right)^{\frac{1}{\alpha+1}} \cdot \left(\frac{\nu(\alpha+1)}{2\alpha}\right)^{\frac{\alpha}{\alpha+1}}$. □

Back to the proof of Theorem 1.3, (5.1) leads to

$$\left(A + \frac{1}{2} + C_{\alpha\tau\nu}\right) \|u^{n+1} - u^n\|_{L^2}^2 + E^{n+1} - E^n \le \|u^{n+1} - u^n\|_{L^2}^2 \left(\|u^n\|_\infty^2 + \frac{1}{2} \|u^{n+1}\|_\infty^2 \right).$$

To prove $E^{n+1} \le E^n$, it suffices to show $A + \frac{1}{2} + C_{\alpha\tau\nu} \ge \frac{3}{2} \max\left\{ \|u^{n+1}\|_\infty^2, \ \|u^n\|_\infty^2 \right\}$. We rewrite the scheme (1.4) as

$$u^{n+1} = \frac{1 + A\tau(-\Delta)^\alpha}{1 + \tau\nu(-\Delta)^{\alpha+1} + A\tau(-\Delta)^\alpha} u^n - \frac{\tau(-\Delta)^\alpha}{1 + \tau\nu(-\Delta)^{\alpha+1} + A\tau(-\Delta)^\alpha} \Pi_N[f(u^n)].$$

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Similarly, we can still apply Lemma 3.1 under the assumption u_0 satisfies zero-mean condition. Recall that

$$\|u^{n+1}\|_\infty \lesssim \|u^{n+1}\|_{\dot{H}^1}\sqrt{\log(\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}+3)}.$$

We will estimate $\|u^{n+1}\|_{\dot{H}^1}$ and $\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}$. As we did in section 3,

$$\begin{cases} \dfrac{1+A\tau|k|^{2\alpha}}{1+A\tau|k|^{2\alpha}+\nu\tau|k|^{2+2\alpha}}\cdot|k|\lesssim|k| \\ \dfrac{\tau|k|^{2\alpha}}{1+A\tau|k|^{2\alpha}+\nu\tau|k|^{2+2\alpha}}\cdot|k|\lesssim\dfrac{\tau}{\tau A}|k|=\dfrac{1}{A}|k|. \end{cases}$$

Hence we derive

$$\|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)}\lesssim\left(1+\frac{1}{A}+\frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1(\mathbb{T}^2)},$$

which is the same argument as before. Similarly, we can derive

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)}\lesssim\left(\frac{1+A\tau}{\nu\tau}+\frac{1}{\nu}\right)(E^n+1).$$

We then prove by induction again.

Step 1: the induction $n\to n+1$ step. Assume $E^n\le E^{n-1}\le\cdots\le E^0$ and $E^n\le\sup_N\ E(\Pi_N u_0)$, we will show $E^{n+1}\le E^n$. This implies $\|u^n\|_{\dot{H}^1}^2=\|\nabla u^n\|_{L^2}^2\le\frac{2E^n}{\nu}\le\frac{2E^0}{\nu}$. By applying the main lemma carefully and $E^0\lesssim E_0+1$,

$$\|u^n\|_\infty^2\lesssim_{E_0}\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)+\nu^{-1}|\log(\tau)|+1.$$

Define $m_0:=\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)$, then the inequality above can be written as

$$\|u^n\|_\infty^2\lesssim_{E_0}m_0+\nu^{-1}|\log(\tau)|+1. \tag{5.2}$$

Similarly,

$$\|u^{n+1}\|_\infty^2\lesssim_{E_0}1+\frac{m_0^3}{A^2}+m_0+\nu^{-3}|\log(\tau)|^3. \tag{5.3}$$

Therefore, we require the following condition:

$$\begin{cases} A+\dfrac{1}{2}+\left(\dfrac{\alpha+1}{\tau}\right)^{\frac{1}{\alpha+1}}\cdot\left(\dfrac{\nu(\alpha+1)}{2\alpha}\right)^{\frac{\alpha+1}{\alpha}}\ge C(E_0)\left(m_0+1+\dfrac{m_0^3}{A^2}+\nu^{-3}|\log(\tau)|^3\right) \\ m_0=\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right). \end{cases}$$

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Now we discuss two cases again:

Case 1: $(\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \geq C(E_0)\nu^{-3}|\log(\tau)|^3$. In this case, it suffices to choose A such that

$$A \gg_{E_0} m_0 = \nu^{-1}\,(1 + \log(A) + |\log(\nu)|).$$

In fact, for $\nu \gtrsim 1$, we can take $A \gg_{E_0} 1$; if $0 < \nu \ll 1$, we will choose $A = C_{E_0} \cdot \nu^{-1}|\log \nu|$, where C_{E_0} is a large constant depending only on E_0. Therefore, in both cases, it suffices to choose

$$A = C_{E_0} \cdot \max\left\{\nu^{-1}|\log(\nu)|,\ 1\right\}.$$

Case 2: $(\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \leq C(E_0)\nu^{-3}|\log(\tau)|^3$. This implies $(\frac{1}{\tau})^{\frac{1}{\alpha+1}} \lesssim (\frac{1}{\nu})^{-4-\frac{1}{\alpha}}$, hence $|\log(\tau)| \lesssim_{E_0} 1 + |\log(\nu)|$ for fixed $0 < \alpha \leq 1$. Now going back to equations (5.2), we have

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0$$

as $\nu^{-1}|\log(\tau)|$ will be dominated by m_0, recall that $m_0 = \nu^{-1}\,(1 + \log(A) + |\log(\nu)|)$. Substituting this new bound to (5.3), we derive that

$$\begin{aligned}
\|u^{n+1}\|_\infty^2 &\lesssim \left(1 + \left(\frac{1 + \|u^n\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)}\right)^2 \\
&\lesssim_{E_0} \left(1 + \left(1 + \frac{m_0}{A}\right)\left(\sqrt{\frac{1}{\nu}}\sqrt{\log\left(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}\right)}\right)\right)^2 \\
&\lesssim_{E_0} \left(1 + \left(1 + \frac{m_0}{A}\right)\sqrt{m_0}\right)^2 \\
&\lesssim_{E_0} 1 + \frac{m_0^3}{A^2} + m_0.
\end{aligned}$$

Thus, it suffices to take

$$A \geq C_{E_0} m_0. \tag{5.4}$$

For the induction base **Step 2**, the proof is exactly the same as in Section 3 and this shows stability of the semi-implicit scheme in the fractional CH case. □

6. Numerical experiments

In this section, we present several numerical results. To begin with we present numerical evidence that show the necessity of the stabilizers.

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TABLE 1. *Choice of centers and radii*

x_i	$-\frac{\pi}{2}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$
y_i	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{3\pi}{4}$	0	$\frac{\pi}{2}$
r_i	$\frac{\pi}{5}$	$\frac{2\pi}{15}$	$\frac{2\pi}{15}$	$\frac{\pi}{10}$	$\frac{\pi}{10}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$

6.1 *A benchmark computation with different A-values*

In this subsection we perform numerical experiments for some benchmark initial data. We first vary the choice of the stabilizers, namely we put $A = 0, 0.01, 1$ for AC and FCH equations. Moreover, we fix the parameters in this subsection as $\nu = 0.01$, $\tau = 0.1$, $N_x = N_y = 256$ and the initial data u_0 is given basically ‘supported’ in seven circles as below:

$$u_0(x,y) = -1 + \sum_{i=1}^{7} f_0\left(\sqrt{(x-x_i)^2 + (y-y_i)^2} - r_i\right), \tag{6.1}$$

where

$$f_0(s) = \begin{cases} 2e^{-\frac{\nu}{s^2}}, & \text{if } s < 0; \\ 0, & \text{otherwise.} \end{cases}$$

The centers and radii of the chosen circles are given in the Table 1 above.

The AC equation: the cases $A = 0$ (left) and 1 (right) can be found in Fig. 1. As you can tell, in practice the usual semi-implicit scheme ($A = 0$) can guarantee the energy dissipation with $\nu = 0.01$, $\tau < 1$ and $\|u_0\|_\infty \leq 1$ already. The patterns obey the curvature motion as desired. The main reason behind this phenomena could be the maximum principle of AC, which is the main difference between AC and CH as mentioned earlier. We refer the readers to the discussion on the effective maximum principles by Li in Li (2021).

The fractional CH equation: the cases $A = 0$ (left) and 0.1 (middle) can be found in Fig. 2. Unlike the AC case, especially for small $\nu = 0.001$, the stabilizer A is necessary if the time step τ is not too small. We see that both cases show instability of the schemes with small A. On the other hand, the case $A = 1$ (right) in Fig. 2 indicates that the patterns follow the Mullins–Sekerka flow as expected.

6.2 *More dynamics and energy evolution with smaller time steps*

In this section, we present more dynamics of the FCH using scheme 1.4 (in comparison with the usual first-order semi-implicit scheme) with smaller time steps. In fact, AC equations are more stable; cf. Fig. 1 below. In Fig. 3, we present the dynamics of 2D fractional CH equation using scheme (1.4), where $\alpha = 0.9$, $\nu = 0.01$, $A = 0$ (the usual semi-implicit method), $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0 = 0.1\sin(x)\sin(y)$. We also present the dynamics with $A = 1$. Comparing these two schemes we see that the stabilized scheme is accurate when the time step is small in a fixed time period. On the other hand, a small time step also allows us to choose small stabilizer A. We refer the readers to Li (2021) and Li (2022b) for more discussion.

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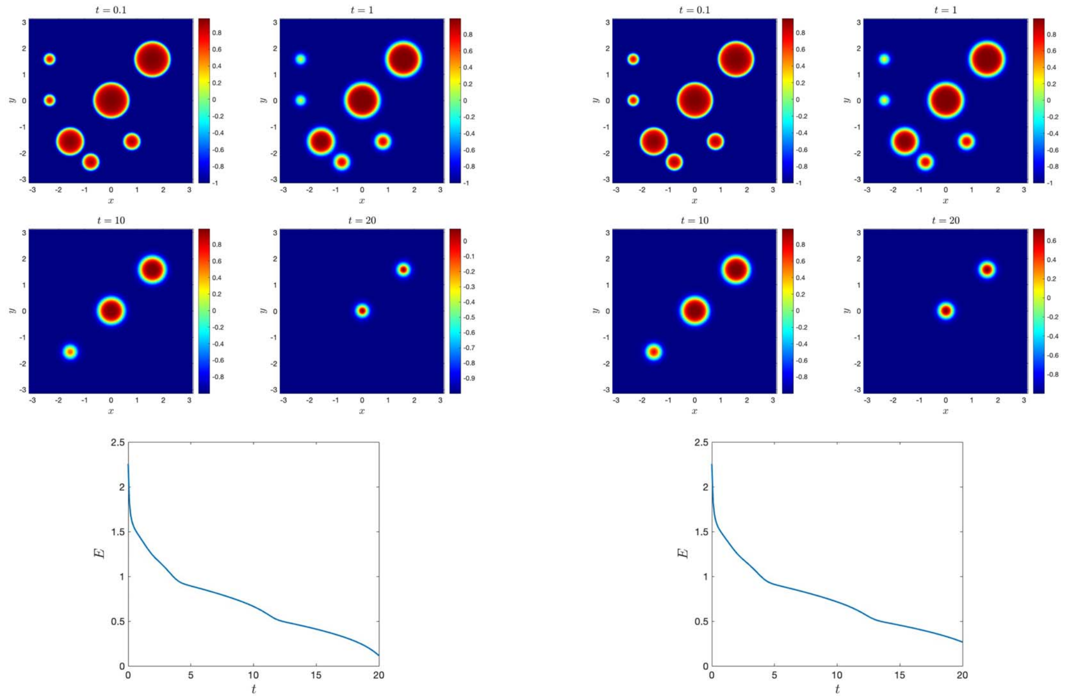

FIG. 1. Dynamics of 2D AC equation using the (stablized) semi-implicit scheme (1.3), where $\nu = 0.01$, $\tau = 0.1$, $N_x = N_y = 256$ and the initial data u_0 is given in (6.1). $A = 0$ (left), $A = 1$ (right). The dynamics are almost the same and energy dissipates in both cases.

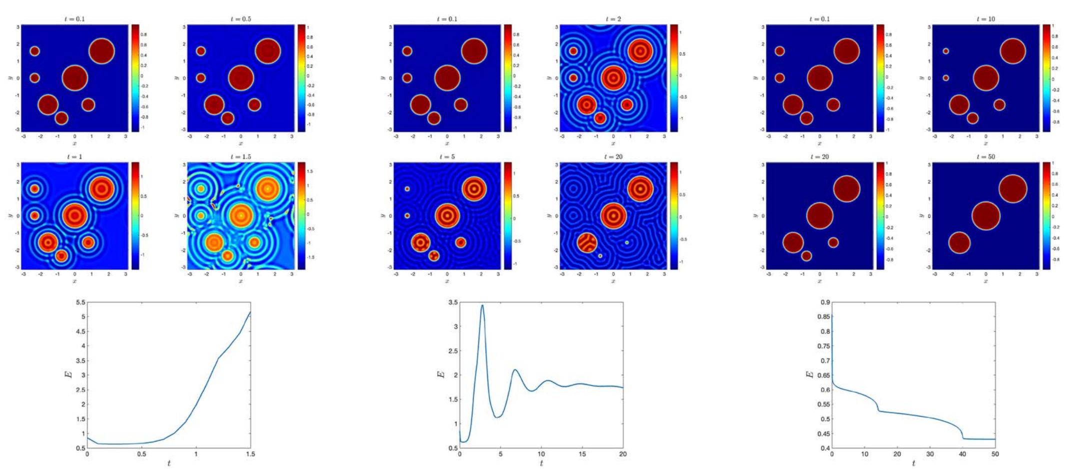

FIG. 2. Dynamics of 2D fractional CH equation using the (stablized) semi-implicit scheme (1.4), where $\alpha = 0.5, \nu = 0.001$, $\tau = 0.1$, $N_x = N_y = 256$ and the initial data u_0 is given in (6.1). $A = 0$ (left), $A = 0.1$ (middle). Both cases show instability of the schemes with small A. The case $A = 1$ (right) indicates energy stability.

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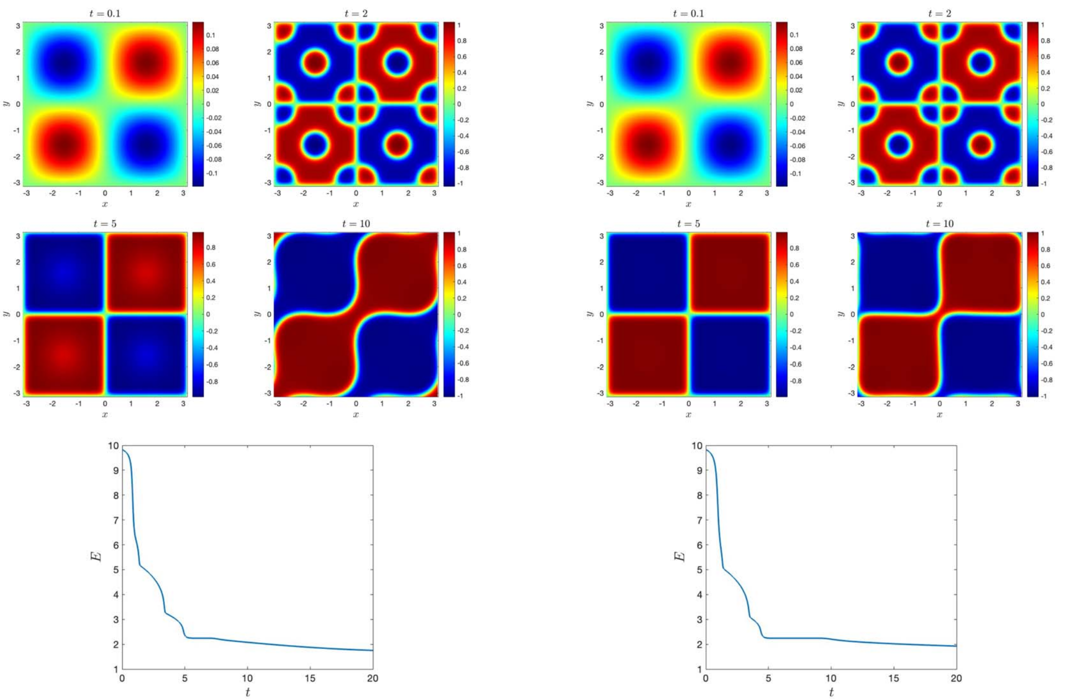

FIG. 3. Dynamics of 2D CH equation using the scheme (1.4), where $\alpha = 0.9$, $\nu = 0.01$, $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0 = 0.1\sin(x)\sin(y)$. $A = 0$ (left), $A = 1$ (right), we see that the patterns are very similar and follow the Mullins–Sekerka flow.

7. Concluding remarks

Throughout this paper, we discussed certain first-order semi-implicit Fourier spectral methods on the AC equation and the fractional CH equation in a two-dimensional torus. We proved the stability (energy decay) of the first-order numerical scheme by adding a stabilizing term $A(u^{n+1}-u^n)$ and $(-\Delta)^\alpha A(u^{n+1}-u^n)$ with a large constant A at least of size $O(\nu^{-1}|\log(\nu)|)$. Note that this stability is preserved independent of time step τ. We also proved an L^2 error estimate between numerical solutions from the semi-implicit scheme and exact solutions.

In the future work, more cases can be discussed on other gradient cases (as mentioned in Remark 1.5) such as general nonlocal AC and CH equations, MBE equations, Cahn–Hilliard–Navier–Stokes system and other equations describing phenomena of interest in material sciences. Higher order schemes and more nonlinear numerical framework will be considered.

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Appendix A. Proof of Lemma 4.1

Proof of Lemma 4.1. We define $f(x,t) = u(x,t)^2$ and $f^\epsilon(x,t) = f(x,t) - \epsilon t$. Since f^ϵ is a continuous function on the compact domain $\mathbb{T}^d \times [0,T]$, it achieves maximum at some point (x_*, t_*), i.e.,

$$\max_{\substack{0\le t\le T,\\ x\in\mathbb{T}^d}} f^\epsilon(x,t) = f^\epsilon(x_*,t_*) := M_\epsilon.$$

We discuss several cases.

`Case 1:` $0 < t_* \le T$ and $M_\epsilon > 1$. This shows $\nabla f^\epsilon(x_*,t_*) = 0,\ \Delta f^\epsilon(x_*,t_*) \le 0$. Note that

$$\nabla f^\epsilon = 2u\nabla u,\ \ \Delta f^\epsilon = 2|\nabla u|^2 + 2u\Delta u,$$

this shows $\nabla u(x_*,t_*) = 0$, $u\Delta u(x_*,t_*) < 0$. However, we also have

$$\begin{aligned}
\partial_t f^\epsilon(x_*,t_*) &= 2u(x_*,t_*)\partial_t u(x_*,t_*) - \epsilon \\
&= 2u(x_*,t_*)(\nu\Delta u(x_*,t_*) - u^3(x_*,t_*) + u(x_*,t_*)) - \epsilon \\
&< -2u^4(x_*,t_*) + 2u^2(x_*,t_*) - \epsilon \\
&< -2\left(u^2(x_*,t_*) - \frac{1}{2}\right)^2 + \frac{1}{2} - \epsilon \\
&< -\epsilon < 0
\end{aligned}$$

as $u^2(x_*,t_*) > 1$ by assumption. This contradicts the hypothesis that f^ϵ achieves its maximum at (x_*,t_*).

`Case 2:` $0 < t_* \le T$ and $M_\epsilon \le 1$. In this case, we obtain

$$\max_{0\le t\le T,\ x\in\mathbb{T}^d} f(x,t) \le 1 + \epsilon T,$$

letting $\epsilon \to 0$, we obtain $f(x,t) \le 1$.

`Case 3:` $t_* = 0$, then

$$\max_{0\le t\le T,\ x\in\mathbb{T}^d} f(x,t) \le \max_{x\in\mathbb{T}^d} f(x,0) + \epsilon T,$$

sending ϵ to 0, we obtain $f(x,t) \le f(x,0)$. This concludes $\|u\|_\infty \le \max\{\|u_0\|\infty,\ 1\}$. □

Appendix B. Proof of Lemma 4.2

Proof of Lemma 4.2. We write the solution u in the mild form

$$u(t) = e^{\nu t\Delta}u_0 + \int_0^t e^{\nu(t-s)\Delta}(u - u^3)\,\mathrm{d}s.$$

We will prove this argument inductively. By previous lemma 4.1, we have $\|u\|_2 \lesssim 1$ as $\|u\|_\infty \lesssim 1$ and we will show $\|u\|_{H^1} \lesssim 1$ for any $t \ge 1$. Then by taking the spatial derivative and L^2 norm in the formula above, we derive

$$\|Du\|_2 \le \|De^{\nu t\Delta}u_0\|_2 + \int_0^t \|De^{\nu(t-s)\Delta}(u - u^3)\|_2\,\mathrm{d}s,$$

where $D^m u$ denotes any differential operator $D^\alpha u$ for any $|\alpha| = m$.

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First, we consider the nonlinear part.

$$\|De^{\nu(t-s)\Delta}(u-u^3)\|_2 \lesssim \|De^{\nu(t-s)\Delta}(u-u^3)\|_\infty \lesssim |K_1 * (u-u^3)|,$$

where K_1 is the kernel corresponding to $De^{\nu(t-s)\Delta}$. Therefore, we estimate that

$$\begin{aligned} |K_1 * (u-u^3)| &\le \|K_1\|_2 \cdot \|u-u^3\|_2 \\ &\lesssim \|K_1\|_2 \cdot \|u\|_2 \end{aligned}$$

by the boundedness of $\|u\|_\infty$. Note that

$$\begin{aligned} \|K_1\|_2 &\lesssim \left(\sum_{k\in\mathbb{Z}^d} |k|^2 e^{-2\nu(t-s)|k|^2} \right)^{\frac{1}{2}} \\ &= \left(\sum_{|k|\ge 1} |k|^2 e^{-2\nu(t-s)|k|^2} \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_1^\infty e^{-2\nu(t-s)r^2} r^{d+1}\,\mathrm{d}r \right)^{\frac{1}{2}}. \end{aligned}$$

The estimates for different dimensions are different. Now we will assume $t \ge 1$ because the other case $t < 1$ is much easier.

`Case 1:` $d = 1$. $\int_1^\infty e^{-2\nu(t-s)r^2} r^2\,\mathrm{d}r \lesssim \frac{e^{-2\nu(t-s)}}{t-s} + \frac{\mathrm{erf}(\sqrt{2\nu(t-s)})}{(t-s)^{3/2}}$, where $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,\mathrm{d}t$, the complementary error function. Letting $\gamma = t-s$,

$$\int_0^t \|De^{\nu(t-s)\Delta}u\|_2\,\mathrm{d}s \lesssim \left(\int_0^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} + \frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}\,\mathrm{d}\gamma \right) \cdot \|u\|_2.$$

For γ small enough, $\frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}$ will dominate the estimate and for γ away from 0, $\frac{e^{-\nu\gamma}}{\gamma^{1/2}}$ shall dominate the estimate. Then we split the integral as below (recall that $t \ge 1$):

$$\begin{aligned} \int_0^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} + \frac{(\mathrm{erf}(\sqrt{\nu\gamma})^{1/2})}{\gamma^{3/4}}\,\mathrm{d}\gamma &\lesssim \int_0^1 \frac{1}{\gamma^{3/4}}\,\mathrm{d}\gamma + \int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\,\mathrm{d}\gamma \\ &\lesssim 1 + \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\,\mathrm{d}\gamma \\ &\lesssim 1. \end{aligned}$$

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`Case 2:` $d = 2$. $\int_1^\infty e^{-2\nu(t-s)r^2} r^3 \,\mathrm{d}r \lesssim \frac{e^{-2\nu(t-s)}}{(t-s)^2} + \frac{e^{-2\nu(t-s)}}{t-s}$. Similar to Case 1, we will split the integral as well. Letting $\gamma = t - s$, we have

$$\begin{aligned}\int_1^t \frac{e^{-\nu\gamma}}{\gamma} + \frac{e^{-\nu\gamma}}{\gamma^{1/2}} \,\mathrm{d}\gamma &\lesssim \int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} \,\mathrm{d}\gamma \\ &\lesssim \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}} \,\mathrm{d}\gamma \\ &\lesssim 1.\end{aligned}$$

However, the estimate in Case 1 does not work for $\gamma \le 1$. Now we estimate $\|K_1 * (u - u^3)\|_{L^2(\mathbb{T}^d)}$ differently. We compute from the Fourier side:

$$\begin{aligned}\|K_1 * (u - u^3)\|_{L^2(\mathbb{T}^d)}^2 &= \sum_{|k|\ge 1} |k|^2 \, e^{-2\nu(t-s)|k|^2} |\widehat{u - u^3}(k)|^2 \\ &\le \max_{|k|\ge 1} \left\{ |k|^2 \, e^{-2\nu(t-s)|k|^2} \right\} \cdot \sum_{|k|\ge 1} |\widehat{u - u^3}(k)|^2 \\ &\lesssim \max_{|k|\ge 1} \left\{ |k|^2 \, e^{-2\nu(t-s)|k|^2} \right\} \cdot \|u\|_{L^2(\mathbb{T}^d)}^2.\end{aligned}$$

Define $g(x) = x^2 \, e^{-2\nu\gamma x^2}$, where $x \ge 0$. Then,

$$g'(x) = 2x \, e^{-2\nu\gamma x^2} \left(1 - 2\nu\gamma x^2\right),$$

which shows the maximum is achieved at $x = \frac{1}{\sqrt{2\nu\gamma}}$ and hence

$$g(x) \le g(\frac{1}{\sqrt{2\nu\gamma}}) \lesssim \frac{1}{\gamma}.$$

Therefore

$$\|De^{\nu(t-s)\Delta}(u - u^3)\|_{L^2(\mathbb{T}^d)} \lesssim \frac{1}{\sqrt{t-s}} \|u\|_{L^2(\mathbb{T}^d)}.$$

Note that this proof works for any dimension. As a result,

$$\int_0^1 \|De^{\nu\gamma\Delta} u\|_2 \,\mathrm{d}\gamma \lesssim \int_0^1 \frac{1}{\sqrt{\gamma}} \,\mathrm{d}\gamma \cdot \|u\|_2 \lesssim 1.$$

This shows $\int_0^t \|De^{\nu(t-s)\Delta} u\|_2 \,\mathrm{d}s \lesssim 1$.

`Case 3:` $d = 3$. As proved in previous case, we will only need to check the case $\gamma \ge 1$. Note that $\int_1^\infty e^{-2\nu\gamma r^2} r^4 \,\mathrm{d}r \lesssim \frac{e^{-2\nu\gamma}}{\gamma}$ for $\gamma \ge 1$. This shows that

$$\begin{aligned}\int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} \,\mathrm{d}\gamma &\lesssim \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}} \,\mathrm{d}\gamma \\ &\lesssim 1.\end{aligned}$$

For the case where $t \le 1$, it is easier because we do not need to split the integral and all integrals from 0 to t can be bounded by the integral from 0 to 1.

Now for the linear part, by Duhamel's Principle, $e^{\nu t\Delta}u_0$ denotes the solution to the heat equation. As is well known, every spatial derivative of the solution $e^{\nu t\Delta}u_0$ solves the heat equation, hence by the energy decay property, we have $\|e^{\nu t\Delta}u_0\|_{H^m} \lesssim \|u_0\|_{H^m}$ for any $1 \le m \le k$. Combining the nonlinear and linear parts, we obtain that $\|u\|_{H^1} \lesssim 1$ independent of $t \ge 0$ and hence $\sup_{t\ge 0}\|u\|_{H^1} \lesssim 1$.

Assume that we have $\sup_{t\ge 0}\|u\|_{H^{m-1}} \lesssim 1$, then the estimate follows by repeating the process above:

$$
\begin{aligned}
\|D(D^{m-1}u)\|_2 &\le \|De^{\nu t\Delta}D^{m-1}u_0\|_2 + \int_0^t \|De^{\nu(t-s)\Delta}D^{m-1}u\|_2 \,\mathrm{d}s \\
&\lesssim \|u_0\|_{H^m} + \int_0^1 \|De^{\nu\gamma\Delta}D^{m-1}u\|_2 \,\mathrm{d}\gamma + \int_1^t \|De^{\nu\gamma\Delta}D^{m-1}u\|_2 \,\mathrm{d}\gamma \\
&\lesssim 1 + \int_0^1 \frac{1}{\sqrt{\gamma}}\,\mathrm{d}\gamma \cdot \|D^{m-1}u\|_2 + \int_0^\infty \frac{e^{-\nu\gamma}}{\sqrt{\gamma}}\,\mathrm{d}\gamma \cdot \|D^{m-1}u\|_2 \\
&\lesssim 1,
\end{aligned}
$$

We finally obtain that

$$
\sup_{t\ge 0}\|u\|_{H^k(\mathbb{T}^d)} \lesssim_k 1. \tag{B.1}
$$

□

Appendix C. Proof of Lemma 4.3

Proof of Lemma 4.3. To simplify the notation, we will use '$\lesssim$' instead of '$\lesssim_{\nu,u_0,A,k}$' in the proof. We will use a similar method to the one provided in Li & Tang (2021).

We can write the scheme as follows:

$$
\begin{aligned}
u^{n+1} &= \frac{1+A\tau}{1+A\tau-\nu\tau\Delta}u^n + \frac{-\tau\Pi_N}{1+A\tau-\nu\tau\Delta}f(u^n) \\
&:= L_1(u^n) + L_2(f(u^n)) \\
&= L_1\left(L_1u^{n-1} + L_2f(u^{n-1})\right) + L_2f(u^n) \\
&= L_1^{m_0+1}u^{n-m_0} + \sum_{l=0}^{m_0} L_1^l L_2 f(u^{n-l}),
\end{aligned} \tag{C.1}
$$

where m_0 will be chosen later. Similar to the continuous version, we prove inductively. To demonstrate the idea, we first show from $\sup_{n\ge 0}\|u^n\|_{H^1(\mathbb{T}^d)} \lesssim 1$ that

$$
\sup_{n\ge 0}\|u^n\|_{H^2(\mathbb{T}^d)} \lesssim 1.
$$

Indeed it suffices to control the $\dot{H}^2$ semi-norm. With no loss, we can assume $\tau < 1$ (or we can assume $\tau < C$ for some harmless constant C). We then discuss two cases:

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`Case 1:` $A\tau \geq \frac{1}{10}$. Then for $0 \neq k \in \mathbb{Z}^d$:

$$\begin{aligned}
\left|\widehat{L_1}(k)\right| &= \frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2} \\
&\leq \frac{11A\tau}{A\tau + \nu\tau|k|^2} \\
&\lesssim \frac{1}{1 + |k|^2},
\end{aligned} \tag{C.2}$$

and

$$\left|\widehat{L_2}(k)\right| = \frac{\tau}{1 + A\tau + \nu\tau|k|^2} \lesssim \frac{1}{1 + |k|^2}. \tag{C.3}$$

We observe that the restriction $A\tau \geq \frac{1}{10}$ is crucial in (C2) and (C3), otherwise the inequalities will depend on the size of τ. To conclude in this case, we first recall that from the energy dissipation $\sup_{n\geq 0} \|u^n\|_{H^1} + \|f(u^n)\|_2 \lesssim 1$. Then we can derive (by choosing $m_0 = n$) that

$$\begin{aligned}
\|u^{n+1}\|_{\dot{H}^2} &\leq \|L_1 u^n\|_{\dot{H}^2} + \|L_2 f(u^n)\|_{\dot{H}^2} \\
&\lesssim \|u^n\|_2 + \|f(u^n)\|_2 \\
&\lesssim 1.
\end{aligned} \tag{C.4}$$

`Case 2:` $A\tau < \frac{1}{10}$. Take m_0 to be the integer such that $\frac{1}{2} \leq m_0\tau < 1$ and thus $m_0 \geq 5$.

$$\begin{aligned}
\left|\widehat{L_1^{m_0+1}}(k)\right| &\leq \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^{m_0+1} \\
&\leq \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^{m_0} \\
&= \left(1 + \frac{\nu\tau|k|^2}{1 + A\tau}\right)^{-m_0}.
\end{aligned}$$

Recall $A\tau < \frac{1}{10} < 1$, then

$$\left(1 + \frac{\nu\tau|k|^2}{1 + A\tau}\right)^{-m_0} \leq \left(1 + \frac{\nu\tau|k|^2}{2}\right)^{-m_0},$$

define $t_0 := m_0\tau$ and we derive

$$\left|\widehat{L_1^{m_0+1}}(k)\right| \leq \left(1 + \frac{1}{2}\nu|k|^2 \frac{t_0}{m_0}\right)^{-m_0}.$$

For any $a > 0$, we consider the function $h(x) = -x \log\left(1 + \frac{a}{x}\right)$, $x > 0$. Then

$$\begin{aligned}
h'(x) &= -\log\left(1 + \frac{a}{x}\right) + \frac{a}{a + x} \\
h''(x) &= \frac{a}{x + a}\left(\frac{1}{x} - \frac{1}{x + a}\right)\ 0.
\end{aligned}$$

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By direct computation, $h(x)$ decreases on $(0,\infty)$. Therefore, recalling $m_0 \geq 5$,

$$\left|\widehat{L_1^{m_0+1}}(k)\right| \leq \left(1+\frac{1}{2}\nu|k|^2\frac{t_0}{m_0}\right)^{-m_0} \leq \left(1+\frac{1}{2}\nu|k|^2\cdot\frac{t_0}{5}\right)^{-5}.$$

As a direct result, we have

$$\begin{aligned}\left|\widehat{L_2}(k)\right|\cdot\sum_{l=0}^{m_0}\left|\widehat{L_1}(k)\right|^l &\leq \left|\widehat{L_2}(k)\right|\cdot\frac{1}{1-\left|\widehat{L_1}(k)\right|}\\ &= \frac{\tau}{1+A\tau+\nu\tau|k|^2}\cdot\frac{1}{1-\frac{1+A\tau}{1+A\tau+\nu\tau|k|^2}}\\ &= \frac{1}{\nu|k|^2}\\ &\lesssim \frac{1}{|k|^2}.\end{aligned}$$

Therefore for $n \geq m_0$,

$$\|u^{n+1}\|_{\dot{H}^2} \lesssim \|u^{n-m_0}\|_2 + \sup_{0\leq l\leq m_0}\|f(u^{n-l})\|_2 \lesssim 1.$$

For $1 \leq n \leq m_0+1$, we apply

$$u^n = L_1^n u^0 + \sum_{l=0}^{n-1} L_1^l L_2 f(u^{n-1-l}).$$

Hence we get

$$\|u^n\|_{\dot{H}^2} \lesssim \|u^0\|_{\dot{H}^2} + \sup_{0\leq l\leq n-1}\|f(u^{n-l-1})\|_2 \lesssim 1.$$

By the energy decay property, the constant depends only on ν, u_0 and A, we can conclude that

$$\sup_{n\geq 0}\|u^n\|_{H^2(\mathbb{T}^d)} \lesssim 1. \tag{C.5}$$

To obtain higher Sobolev H^k norm control, we can repeat the bootstrap process above and derive the desired result. Inductively, we can derive

$$\begin{cases}\|u^{n+1}\|_{\dot{H}^m} \lesssim \|u^n\|_{\dot{H}^{m-2}} + \|f(u^n)\|_{\dot{H}^{m-2}},\ A\tau \geq \dfrac{1}{10};\\ \|u^{n+1}\|_{\dot{H}^m} \lesssim \|u^{n-m_0}\|_{\dot{H}^{m-2}} + \sup\limits_{0\leq l\leq m_0}\|f(u^{n-l})\|_{\dot{H}^{m-2}},\ A\tau < \dfrac{1}{10},\ n \geq m_0;\\ \|u^n\|_{\dot{H}^m} \lesssim \|u^0\|_{\dot{H}^m} + \sup\limits_{0\leq l\leq n-1}\|f(u^{n-l-1})\|_{\dot{H}^{m-2}},\ A\tau < \dfrac{1}{10},\ n \leq m_0+1.\end{cases}$$

□

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