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Non-uniqueness of Steady-State Weak Solutions to the Surface Quasi-Geostrophic Equations

Xinyu Cheng[1], **Hyunju Kwon**[2], **Dong Li**[3]

[1] School of Mathematical Sciences, Fudan University, Shanghai, China. E-mail: xycheng@fudan.edu.cn
[2] School of Mathematics, Institute for Advanced Study, Princeton, USA. E-mail: hkwon@ias.edu
[3] SUSTech International Center for Mathematics and Department of Mathematics, Southern University of Science and Technology, Shenzhen, China. E-mail: lid@sustech.edu.cn

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Abstract: We show the existence of nontrivial stationary weak solutions to the surface quasi-geostrophic equations on the two dimensional periodic torus.

1. Introduction

Consider the two dimensional dissipative surface quasi-geostrophic (SQG) equations for $\theta = \theta(x,t) : \mathbb{T}^2 \times [0,\infty) \to \mathbb{R}$:

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta = -\nu \Lambda^\gamma \theta, \quad \text{in} \quad \mathbb{T}^2 \times (0,\infty); \\ u = \nabla^\perp \Lambda^{-1}\theta = (-\partial_2 \Lambda^{-1}\theta, \partial_1 \Lambda^{-1}\theta) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta); \\ \theta|_{t=0} = \theta_0, \end{cases} \tag{SQG}$$

where $\nu \geq 0$ is the viscosity, $0 < \gamma \leq 2$ and $\mathbb{T}^2 = [-\pi,\pi]^2$ is the periodic torus. Here the unknown scalar function θ denotes the potential temperature in the context of geophysical fluid dynamics [8,13]. This transport equation models the evolution of the temperature in a fast rotating stratified fluid and can be derived from a more complete 3D system via Boussinesq approximation [13]. In Eq. (SQG), $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ is the pair of Riesz transforms and $\nabla^\perp = (-\partial_2, \partial_1)$. For $s \geq 0$ the fractional Laplacian $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ is defined by (under suitable assumptions on θ) $\widehat{\Lambda^s\theta}(k) = |k|^s\hat{\theta}(k)$ for $k \in \mathbb{Z}^2$. For negative s the formula is restricted to nonzero wave numbers. We consider solutions with zero mean, i.e. $\int_{\mathbb{T}^2} \theta(x,t)\,dx = 0$, which is invariant under the dynamics thanks to incompressibility. The purpose of this work is to construct stationary weak solutions to (SQG). By using integration by parts, one way to define stationary weak solutions to (SQG) is to drop the $\partial_t\theta$ term and require

$$-\int_{\mathbb{T}^2} \theta u \cdot \nabla\phi\,dx = -\nu \int_{\mathbb{T}^2} \theta\Lambda^\gamma\phi\,dx, \quad \forall\phi \in C^\infty(\mathbb{T}^2). \tag{1.1}$$

However, this definition requires the strong assumption $\theta \in L^2$ which did not take into account of the incompressibility condition. On the other hand, it is possible to define stationary weak solutions using the mere $\dot{H}^{-\frac{1}{2}}$-regularity. The starting point is to note that the operators $\mathcal{R}_j$, $j = 1, 2$ are skew-symmetric, i.e. $\langle \mathcal{R}_j f, g\rangle = -\langle f, \mathcal{R}_j g\rangle$ where $\langle , \rangle$ denotes the usual L^2 (real) inner product. Using this one can derive for $\theta \in L^2$ (below $[A, B] = AB - BA$ is the usual commutator):

$$\langle \theta \mathcal{R}_j \theta, \phi\rangle = -\frac{1}{2}\langle \theta, [\mathcal{R}_j, \phi]\theta\rangle, \quad \forall \phi \in C^\infty(\mathbb{T}^2).$$

Since $\|[R_j, \phi]\theta\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\phi\|_{H^3}\|\theta\|_{\dot{H}^{-\frac{1}{2}}}$ (see Proposition 5.1), it is then not difficult to see that $\dot{H}^{-\frac{1}{2}}$-regularity suffices for defining a stationary weak solution.

Definition 1.1. We say $\theta \in \dot{H}^{-\frac{1}{2}}(\mathbb{T}^2)$ with zero mean is a stationary weak solution to (SQG) if

$$\frac{1}{2}\int_{\mathbb{T}^2} (\Lambda^{-\frac{1}{2}}\theta) \cdot \Lambda^{\frac{1}{2}}([\mathcal{R}^\perp, \nabla\psi]\theta)dx = -\nu \int_{\mathbb{T}^2} (\Lambda^{-\frac{1}{2}}\theta)\Lambda^{\gamma+\frac{1}{2}}\psi dx, \quad \forall \psi \in C^\infty(\mathbb{T}^2),$$

where $[\mathcal{R}^\perp, \nabla\psi]\theta = -[\mathcal{R}_2, \partial_1\psi]\theta + [\mathcal{R}_1, \partial_2\psi]\theta$.

In the non-steady case, weak solutions in $L^2_{t,\mathrm{loc}}\dot{H}_x^{-\frac{1}{2}}$ can be defined similarly by employing time-dependent test functions. Resnick [14] proved the global existence of a weak solution to (SQG) for $\nu \geq 0$ and $0 < \gamma \leq 2$ in $L_t^\infty L_x^2$ for any initial data $\theta_0 \in L_x^2(\mathbb{T}^2)$. Marchand [12] obtained a global weak solution in $L_t^\infty H_x^{-\frac{1}{2}}$ for $\theta_0 \in \dot{H}_x^{-\frac{1}{2}}(\mathbb{R}^2)$ or $L_t^\infty L_x^p$ for $\theta_0 \in L_x^p(\mathbb{R}^2)$, $p \geq \frac{4}{3}$, when $\nu > 0$ and $0 < \gamma \leq 2$. Note that in Marchand's result, the inviscid case $\nu = 0$ requires $p > 4/3$ since the embedding $L^{\frac{4}{3}} \hookrightarrow \dot{H}^{-\frac{1}{2}}$ is not compact, whereas for the diffusive case one has extra $L_t^2\dot{H}^{\frac{\gamma}{2}-\frac{1}{2}}$ a prior control given by the energy identity.

For non-stationary smooth solutions with zero mean, one has conservation ($\nu = 0$) or dissipation ($\nu > 0$) of $\dot{H}^{-\frac{1}{2}}$-Hamiltonian. Indeed for $\nu = 0$ by using the identity (below $P_{<J}$ is a smooth frequency projection to $\{|k| \leq \text{constant} \cdot 2^J\}$)

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\frac{1}{2}}P_{<J}\theta\|_2^2 = -\int P_{<J}(\theta\mathcal{R}^\perp\theta) \cdot P_{<J}\mathcal{R}\theta dx,$$

one can prove the conservation of $\|\Lambda^{-\frac{1}{2}}\theta\|_2^2$ under the assumption $\theta \in L^3_{t,x}$ (see also [10]). We also mention that for the non-dissipative case in the positive direction uniqueness of SQG patches with moving boundary satisfying the arc-chord condition was obtained in recent [4].

In this paper, we prove the non-uniqueness of stationary weak solutions to (SQG).

Theorem 1.2. *For any $\nu \geq 0$, $\gamma \in (0, \frac{3}{2})$, and $\frac{1}{2} \leq \alpha_* < \frac{1}{2} + \min(\frac{1}{6}, \frac{3}{2} - \gamma)$, there exist infinitely many stationary weak solutions θ to (SQG) with zero mean satisfying $\Lambda^{-1}\theta \in C^{\alpha_*}(\mathbb{T}^2)$.*

Remark 1.3. The restriction $\gamma < \frac{3}{2}$ in Theorem 1.2 can be seen by a crude heuristic using the plane wave ansatz localized around frequency λ. The domination of nonlinearity versus dissipation yields $\|\Lambda^{-1}\theta\|_\infty \gg \lambda^{\gamma-2}$. The Hölder regularity of $\Lambda^{-1}\theta$ yields $\|\Lambda^{-1}\theta\|_\infty \lesssim \lambda^{-\alpha_*}$ where $\alpha_* > \frac{1}{2}$. Thus $\gamma \leq 2 - \alpha_* < \frac{3}{2}$.

The convex integration scheme developed in the seminal works [6,7] can be applied to general active scalar models such as $\partial_t\theta + \nabla\cdot(\theta u) = 0$ where $\widehat{u} = m(k)\widehat{\theta}(k)$ and $m(k)$ is a general Fourier multiplier. By using a plane wave ansatz $\theta = a_k e^{i\lambda k\cdot x} + a_k^* e^{-i\lambda k\cdot x}$ with $|k| = 1$ and $\lambda \gg 1$, one can extract the non-oscillatory part of $\nabla\cdot(\theta u)$ as $\nabla\cdot(|a_k|^2(m(-\lambda k) + m(\lambda k)))$ which vanishes if m is odd. This is known as the odd multiplier obstruction [5,10,15]. Previously the non-uniqueness results were established only for active scalar equations with non-odd multipliers [10,15]. In [1] this issue was resolved for the time-dependent SQG, by using the momentum equation[1] for $v = \Lambda^{-1}u$ and rewriting the nonlinearity $u\cdot\nabla v - (\nabla v)^T\cdot u$ as the sum of a divergence of a 2-tensor, and a gradient of a scalar function. In particular, weak solutions $\Lambda^{-1}\theta \in C_t^\sigma C_x^\beta$, $\frac{1}{2} < \beta < \frac{4}{5}$, $\sigma < \frac{\beta}{2-\beta}$, with any prescribed energy $\|\Lambda^{-\frac{1}{2}}\theta(t)\|_2 = e(t) \in C_c^\infty$ were constructed when $\nu \geq 0$, $0 < \gamma < 2-\beta$. Note that the restriction $\beta - 1 < 1-\gamma$ accords with the critical $\|\theta\|_{L_t^\infty \dot{C}^{1-\gamma}}$ norm. Recently Isett and Ma [9] give another direct approach at the level of θ. For some more recent application of convex integration to other fluid models, see [2,3,11] and the references therein.

The modest goal of this work is to introduce another approach[2] to overcome the odd multiplier obstruction by working directly with the scalar function $f = \Lambda^{-1}\theta$ and developing a concise framework tailor-made for similar problems. From our analysis it appears that the indirect momentum formulation emphasized in [1] can be circumvented and one can settle the problem directly using the special structure of SQG. Returning to the plane wave ansatz, a decisive step for the SQG nonlinearity is to identify the nontrivial non-oscillatory part after removing the $\nabla^\perp$-direction. More precisely, consider $f = \sum_l a_l(x)\cos(\lambda l\cdot x)$ where $|l| = 1$ and $\lambda \gg 1$, then (see Lemma 2.1)

$$\Lambda f = \sum_l \Big(\lambda f + (l\cdot\nabla)a_l \sin(\lambda l\cdot x) + (T_{\lambda l}^{(1)} a_l)\cos(\lambda l\cdot x) + (T_{\lambda l}^{(2)} a_l)\sin(\lambda l\cdot x)\Big).$$

By a short computation we arrive at

$$\Lambda f \nabla^\perp f \overset{\circ}{\approx} -\frac{1}{4}\lambda \sum_l (l\cdot\nabla)(a_l^2) l^\perp + \text{error terms},$$

where the notation $\overset{\circ}{\approx}$ is defined in (1.2). We then use a novel algebraic lemma (Lemma 2.2) to obtain nontrivial projection in the gradient direction. One should note that in the above computation, the leading $O(\lambda^2)$ term vanishes which completely accords with the odd multiplier obstruction problem mentioned earlier. What is remarkable is that in the next $O(\lambda)$ term there is nontrivial non-oscillatory contribution coming from the commutator piece $[\Lambda, a_l]\cos\lambda x$. This seems to be the crucial technical difference between SQG and Euler.

Our next result is about the weak rigidity of solutions in the time-dependent case. It improves Theorem 1.3 of [10] all the way from $L_t^p L_x^2$, $p > 2$ to $L_t^2 \dot{H}^{-\frac{1}{2}+}$. The proof can be found in Sect. 5.

Theorem 1.4 (Weak rigidity). *Let $\nu \geq 0$ and $0 < \gamma \leq 2$. Suppose $f = \lim_n \theta_n$ is a weak limit of solutions (SQG) in $L_t^2 \dot{H}^s$ for $s > -\frac{1}{2}$. Then f must also be a weak solution.*

[1] This approach originates from an exposition in [17], which dates back to Resnick's thesis [14].

[2] Our work was first completed in the summer of 2018 when all three authors were at UBC and we thank the mathematics department for its hospitality.

Notation. For any two quantities A and B, $A \lesssim B$ denotes $A \le CB$ for some absolute constant $C > 0$. Similarly, $A \gtrsim B$ means $A \ge CB$, and $A \sim B$ when $A \lesssim B$ and $A \gtrsim B$. For a real number X, we use $X+$ for $X + \epsilon$ when $\epsilon > 0$ is sufficiently small. For example the space L^{2+} means $L^{2+\epsilon}$ for some $\epsilon > 0$ sufficiently small. For any two vector functions v and w, we denote

$$\boxed{v \overset{\circ}{\approx} w, \quad \text{if} \quad v = w + \nabla^{\perp} p} \tag{1.2}$$

holds for some smooth scalar function p. The mean of f on $\mathbb{T}^2$ is denoted by $\overline{f} = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(x)dx$. We define the function space $C_0^{\infty}(\mathbb{T}^2)$ as

$$C_0^{\infty}(\mathbb{T}^2) = \left\{ f \in C^{\infty}(\mathbb{T}^2) : \overline{f} = 0 \right\}. \tag{1.3}$$

For any $1 \le p \le \infty$, we denote $\|f\|_p = \|f\|_{L^p(\mathbb{T}^2)}$ as the usual Lebesgue norm. For f on $\mathbb{T}^2$, we follow the Fourier transform convention $\hat{f}(k) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(x)e^{-ix\cdot k}dx$ and $f(x) = \sum_{k\in\mathbb{Z}^2} \hat{f}(k)e^{ik\cdot x}$. The convolution operation $*$ is defined by $(f * g)(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(x-y)g(y)dy$, which implies $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$ and $\widehat{fg}(k) = \sum_{l\in\mathbb{Z}^2} \hat{f}(l)\hat{g}(k-l)$.

For $s \in \mathbb{R}$, the homogeneous $\dot{H}^s$-Sobolev norm is defined by $\|f\|_{\dot{H}^s(\mathbb{T}^2)} = \left(\sum_{0\ne k\in\mathbb{Z}^2} |k|^{2s}|\hat{f}(k)|^2\right)^{\frac{1}{2}}$.

Parameters. Throughout this paper, we fix parameters as follows. $\nu \ge 0$, $0 < \gamma < \frac{3}{2}$,

$$\lambda_n = \left\lceil \lambda_0^{b^n} \right\rceil, \quad r_n = \lambda_n^{-\beta}, \quad \mu_{n+1} = (\lambda_{n+1}\lambda_n)^{\frac{1}{2}}, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.4}$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Here we first choose $0 < \beta < \min(\frac{1}{3}, 3-2\gamma)$ to satisfy $\frac{1+\beta}{2} > \alpha_*$ where the prescribed regularity index α_* was specified in Theorem 1.2. We then choose $b - 1 \in (0, b_0)$ so that $\alpha = \frac{1}{2} + \frac{\beta}{2b} - (b-1)^3 \ge \alpha_*$ where b_0 is defined in Proposition 3.1. Lastly λ_0 was chosen sufficiently large according to Proposition 3.1.

See also Appendix 5 for more explicit dependence of constants and the rationale for the specific choices.

2. Construction of the Perturbation

For $f = \Lambda^{-1}\theta$ the steady-state SQG equation is $\nabla \cdot \left(\Lambda f \nabla^{\perp} f\right) = -\nu\Lambda^{\gamma+1} f$ which follows from $\Lambda f \nabla^{\perp} f \overset{\circ}{\approx} \nu\Lambda^{\gamma-1}\nabla f$. The idea is to find approximate solutions $(f_{\le n}, q_n) \in C_0^{\infty}(\mathbb{T}^2) \times C_0^{\infty}(\mathbb{T}^2)$ solving the relaxed equation

$$\Lambda f_{\le n}\nabla^{\perp} f_{\le n} \overset{\circ}{\approx} \nu\Lambda^{\gamma-1}\nabla f_{\le n} + \nabla q_n, \tag{2.1}$$

such that $q_n \to 0$ in the limit. This will be done inductively.

Writing $f_{\le n+1} = f_{\le n} + f_{n+1}$, we first show that for given q_n one can solve

$$\Lambda f_{n+1}\nabla^{\perp} f_{n+1} + \nabla q_n \overset{\circ}{\approx} \text{small error}, \tag{2.2}$$

where the left hand side is the main piece in

$$(\Lambda f_{n+1}\nabla^{\perp} f_{n+1} + \nabla q_n) + \Lambda f_{\le n}\nabla^{\perp} f_{n+1} + \Lambda f_{n+1}\nabla^{\perp} f_{\le n} \overset{\circ}{\approx} \nabla q_{n+1} + \nu\Lambda^{\gamma-1}\nabla f_{n+1}. \tag{2.3}$$

2.1. Derivation of the leading order part. Consider the ansatz ($f = f_{n+1}$)

$$f(x) = \sum_l a_l(x)\cos(\lambda l\cdot x), \tag{2.4}$$

where the frequency of a_l is much smaller than λ and the summation over l is finite.

Lemma 2.1 (Leibniz). *Let* $|l| = 1$, $\lambda l \in \mathbb{Z}^2$, *and* $g(x) = a(x)\cos(\lambda l\cdot x)$. *Then,*

$$\Lambda g = \lambda g + (l\cdot\nabla a)\sin(\lambda l\cdot x) + (T^{(1)}_{\lambda l}a)\cos(\lambda l\cdot x) + (T^{(2)}_{\lambda l}a)\sin(\lambda l\cdot x),$$

where

$$\widehat{T^{(1)}_{\lambda l}a}(k) = \left(\frac{|\lambda l + k| + |\lambda l - k|}{2} - \lambda\right)\widehat{a}(k),$$
$$\widehat{T^{(2)}_{\lambda l}a}(k) = i\left(\frac{|\lambda l + k| - |\lambda l - k|}{2} - l\cdot k\right)\widehat{a}(k). \tag{2.5}$$

Proof. We begin with the following simple fact: if $\widehat{T_m g}(k) = m(k)\widehat{g}(k)$, then $\forall n \in \mathbb{Z}^2$, $T_m(g(x)e^{in\cdot x}) = (T_{m_1}g)e^{in\cdot x}$, where $m_1(k) = m(k+n)$. Noting that $\widehat{\Lambda g}(k) = |k|\widehat{g}(k)$, we have

$$\Lambda(a(x)\cos(\lambda l\cdot x)) = \frac{1}{2}\Lambda(a(x)e^{i\lambda l\cdot x}) + \frac{1}{2}\Lambda(a(x)e^{-i\lambda l\cdot x}) = \frac{1}{2}\Lambda_{m_1}(a)e^{i\lambda l\cdot x} + \frac{1}{2}\Lambda_{m_2}(a)e^{-i\lambda l\cdot x},$$

where $\widehat{\Lambda_{m_1}a}(k) = |k+\lambda l|$ and $\widehat{\Lambda_{m_2}a}(k) = |k-\lambda l|$. The desired identity then follows rearranging terms. □

By using Lemma 2.1, we have

$$\Lambda f\nabla^{\perp} f \overset{\circ}{\approx} \boxed{\text{main}} + \boxed{\text{non-oscillatory error}} + \boxed{\text{oscillatory error}}, \tag{2.6}$$

where (below $l^{\perp} = (-l_2, l_1)^{\mathsf{T}}$ for $l = (l_1, l_2)^{\mathsf{T}}$)

$$\boxed{\text{main}} = -\frac{1}{4}\lambda\sum_l (l\cdot\nabla)(a_l^2)l^{\perp},$$

$$\boxed{\text{non-oscillatory error}} = -\frac{1}{2}\lambda\sum_l (T^{(2)}_{\lambda l}a_l)a_l l^{\perp} + \frac{1}{2}\sum_l (T^{(1)}_{\lambda l}a_l)\nabla^{\perp}a_l,$$

$$\boxed{\text{oscillatory error}} = \frac{1}{2}\sum_l (l\cdot\nabla a_l + T^{(2)}_{\lambda l}a_l)(\lambda a_l l^{\perp}\cos(2\lambda l\cdot x) + \nabla^{\perp}a_l\sin(2\lambda l\cdot x)) \tag{osc1}$$

$$-\frac{1}{2}\sum_l (T^{(1)}_{\lambda l}a_l)(\lambda a_l l^{\perp}\sin(2\lambda l\cdot x) - \nabla^{\perp}a_l\cos(2\lambda l\cdot x)) \tag{osc2}$$

$$-\lambda \sum_{l\neq l'} (l\cdot\nabla a_l + T^{(2)}_{\lambda l} a_l) a_{l'} (l')^{\perp} \sin(\lambda l\cdot x)\sin(\lambda l'\cdot x) \tag{osc3}$$

$$+ \sum_{l\neq l'} (l\cdot\nabla a_l + T^{(2)}_{\lambda l} a_l) \nabla^{\perp} a_{l'} \sin(\lambda l\cdot x)\cos(\lambda l'\cdot x) \tag{osc4}$$

$$-\lambda \sum_{l\neq l'} (T^{(1)}_{\lambda l} a_l) a_{l'} (l')^{\perp} \cos(\lambda l\cdot x)\sin(\lambda l'\cdot x) \tag{osc5}$$

$$+ \sum_{l\neq l'} (T^{(1)}_{\lambda l} a_l) \nabla^{\perp} a_{l'} \cos(\lambda l\cdot x)\cos(\lambda l'\cdot x). \tag{osc6}$$

Note that the leading-order term $\lambda f \nabla^{\perp} f$ in $\Lambda f \nabla^{\perp} f$ vanishes since $\nabla^{\perp}\left(\frac{\lambda}{2} f^2\right) \overset{\circ}{\approx} 0$.

2.2. Matching. We begin with a simple yet powerful lemma.

Lemma 2.2 (Algebraic Lemma). *For a given $Q \in C_0^{\infty}(\mathbb{T}^2)$, we have the decomposition identity*

$$\sum_{j=1}^{2} l_j^{\perp} (l_j\cdot\nabla)(\mathcal{R}_j^o Q) \overset{\circ}{\approx} \nabla Q,$$

where $l_1 = (\frac{3}{5}, \frac{4}{5})^{\mathsf{T}}$, $l_2 = (1,0)^{\mathsf{T}}$, and the Riesz-type transforms $\mathcal{R}_j^o$, $j = 1, 2$ are defined by

$$\widehat{\mathcal{R}_1^o}(k_1,k_2) = \frac{25(k_2^2-k_1^2)}{12|k|^2}, \quad \widehat{\mathcal{R}_2^o}(k_1,k_2) = \frac{7(k_2^2-k_1^2)}{12|k|^2} + \frac{4k_1k_2}{|k|^2}. \tag{2.7}$$

Proof. This follows from the identity $\sum_{j=1}^{2} (l_j^{\perp}\cdot\nabla)(l_j\cdot\nabla)(\mathcal{R}_j^o Q) = \Delta Q$. □

Proposition 2.3. *Set l_j and $\mathcal{R}_j^o$, $j = 1, 2$ as in Lemma 2.2. For given $q_n \in C_0^{\infty}(\mathbb{T}^2)$, choose $C_0 \geq 2$ to be a fixed constant and*

$$a_{j,n+1}^{\text{perfect}} = 2\sqrt{\frac{r_n}{5\lambda_{n+1}}}\sqrt{C_0 + \mathcal{R}_j^o \frac{q_n}{r_n}}, \tag{2.8}$$

where (λ_{n+1}, r_n) are taken as in (1.4). Then

$$-\frac{1}{4}\cdot(5\lambda_{n+1})\cdot\left(\sum_{j=1}^{2} l_j^{\perp}(l_j\cdot\nabla)(a_{j,n+1}^{\text{perfect}})^2\right) + \nabla q_n \overset{\circ}{\approx} 0. \tag{2.9}$$

Proof. The proof follows from applying Lemma 2.2 to $Q = q_n$. Note here we choose the specific form $5\lambda_{n+1}$ with $\lambda_{n+1} \in \mathbb{N}$ for the following technical reason: in Lemma 2.1, we need $\lambda l \in \mathbb{Z}^2$; in Lemma 2.2, $l_1 = (\frac{3}{5}, \frac{4}{5})^{\mathsf{T}}$; clearly we need to choose $\lambda = 5\lambda_{n+1} \in 5\mathbb{N}$. □

We now choose

$$f_{n+1}(x) = \sum_{j=1}^{2} a_{j,n+1}(x)\cos(5\lambda_{n+1} l_j\cdot x), \quad a_{j,n+1} = P_{\leq \mu_{n+1}} a_{j,n+1}^{\text{perfect}}, \tag{2.10}$$

where $\widehat{P_{\leq\mu_{n+1}} g}(k) = \psi(\frac{k}{\mu_{n+1}})\widehat{g}(k)$, and $\psi \in C_c^{\infty}(\mathbb{R}^2)$ satisfies $\psi(k) = 0$ for $|k| \geq 1$, and $\psi(k) = 1$ for $|k| \leq \frac{1}{2}$. We have $\Lambda f_{n+1}\nabla^{\perp} f_{n+1} + \nabla q_n \overset{\circ}{\approx}$ small error. In the next section we estimate the errors.

3. Error Estimates

In this section we prove the following proposition which is the key in the whole iteration procedure.

Proposition 3.1. *Given $\nu \geq 0$, $0 < \gamma < \frac{3}{2}$ and $0 < \beta < \min\left(\frac{1}{3}, 3 - 2\gamma\right)$, there exists $b_0 = b_0(\nu, \gamma, \beta)$ such that for any $0 < b - 1 < b_0$ we can find $\Lambda_0 = \Lambda_0(\nu, \gamma, \beta, b)$ for which the following holds. If $\lambda_0 \geq \Lambda_0$ and $(f_{\leq n}, q_n)$ satisfies*

- *the frequencies of $f_{\leq n}$ and q_n are localized to $\leq 6\lambda_n$ and $\leq 12\lambda_n$, respectively,*
- *$\|f_{\leq n}\|_{C^\alpha(\mathbb{T}^2)} \leq 100$ and $\|q_n\|_X \leq r_n$ where*

$$\alpha = \frac{1}{2} + \frac{\beta}{2b} - (b-1)^3;$$

$$\|q\|_X := \|q\|_\infty + \sum_{j=1}^{2} \|\mathcal{R}_j^o q\|_\infty, \tag{3.1}$$

and $\mathcal{R}_j^o$ is defined in (2.7). Then there exists $q_{n+1} \in C_0^\infty(\mathbb{T}^2)$ solving (2.3) with frequency localized to $\leq 12\lambda_{n+1}$, f_{n+1} defined by (2.10) satisfying

$$\|q_{n+1}\|_X \leq r_{n+1}. \tag{3.2}$$

We now explain the motivation for choosing the X-norm in (3.1). First of all, $q = q_n$ represents the residual error at step n and in the Hölderian context an ideal choice is to use $\|q\|_\infty$ only. However, there are Riesz-type operators $\mathcal{R}_j^o$, $j = 1, 2$ which appear somewhat inevitably in the "matching" process (see for example Proposition 2.3 and especially (2.8)). For this reason it is necessary to include $\|\mathcal{R}_j^o q\|_\infty$ in the working X-norm. To prove Proposition 3.1, we need several technical lemmas.

Lemma 3.2. *Suppose $a : \mathbb{T}^2 \to \mathbb{R}$, $a \in L^\infty(\mathbb{T}^2)$ such that $\overline{a} = 0$ with $\mathrm{supp}(\widehat{a}) \subset \{|k| \leq \mu\}$ and $\mu \geq 10$. Let $m \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ be a homogeneous function of degree 0 and T_m is the Fourier multiplier defined by $\widehat{T_m f}(k) = m(k)\widehat{f}(k)$, then we have $\|T_m a\|_\infty \lesssim \|a\|_\infty \log \mu$. Here the implied constant depends on m.*

Proof. Using the Littlewood–Paley decomposition [16], splitting into low and high frequencies and choosing integer $J \sim 2\log\mu$, we obtain

$$\begin{aligned} \|T_m a\|_\infty &\lesssim (J+3)\|a\|_\infty + 2^{-J}\|\nabla a\|_\infty \\ &\lesssim (J+3+2^{-J}\mu)\|a\|_\infty \lesssim \|a\|_\infty \log\mu. \end{aligned}$$

□

We now state two useful facts. Assume $f \in C^\infty(\mathbb{T}^2)$ and $K \in L^1(\mathbb{R}^2)$ with $m(\xi) = \int_{\mathbb{R}^2} K(z) e^{-i\xi\cdot z} dz$. Then[3]

$$(T_m f)(x) := \sum_k m(k)\hat{f}(k)e^{ik\cdot x} = \int_{\mathbb{R}^2} K(z) f(x-z)dz, \tag{3.3}$$

$$\|T_m f\|_{L_x^p(\mathbb{T}^2)} \leq \|K\|_{L_x^1(\mathbb{R}^2)} \|f\|_{L_x^p(\mathbb{T}^2)}, \quad \forall\, 1 \leq p \leq \infty. \tag{3.4}$$

[3] Here and below we still denote by f its periodic extension to all of $\mathbb{R}^2$.

Assume $f, g \in C^\infty(\mathbb{T}^2)$ and $K \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ with $m(\xi,\eta) = \int_{\mathbb{R}^2\times\mathbb{R}^2} K(z_1,z_2) e^{-i\xi\cdot z_1 - i\eta\cdot z_2} dz_1 dz_2$. Then

$$T_m(f,g)(x) := \sum_k \Big(\sum_{k'\in\mathbb{Z}^2} m(k', k-k') \hat{f}(k')\hat{g}(k-k') \Big) e^{ik\cdot x} \tag{3.5}$$

$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} K(z_1,z_2) f(x-z_1) g(x-z_2) dz_1 dz_2, \tag{3.6}$$

and consequently $\|T_m(f,g)\|_{L^r_x(\mathbb{T}^2)} \le \|K\|_{L^1_x(\mathbb{R}^2\times\mathbb{R}^2)} \|f\|_{L^p_x(\mathbb{T}^2)} \|g\|_{L^q_x(\mathbb{T}^2)}$ for any $1 \le r, p, q \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Lemma 3.3. *Assume* $b_0 : \mathbb{T}^2 \to \mathbb{R}$ *with* $\mathrm{supp}(\widehat{b_0}) \subset \{|k| \le \mu\}$ *and* $10 \le \mu \le \frac{1}{2}\lambda$. *Then* (*see* (2.5))

$$\|T^{(1)}_{\lambda l} b_0\|_\infty \lesssim \lambda^{-1}\mu^2 \|b\|_\infty, \quad \|T^{(2)}_{\lambda l} b_0\|_\infty \lesssim \lambda^{-2}\mu^3 \|b_0\|_\infty,$$
$$\|\Delta^{-1}\nabla T^{(2)}_{\lambda l} b_0\|_X \lesssim \|b_0\|_\infty \lambda^{-2}\mu^2 \log \mu.$$

Proof. We show only the first one as the rest are similar. Choose $\phi_1 \in C_c^\infty(\mathbb{R}^2)$ such that $\phi_1(\xi) \equiv 1$ for $|\xi| \le 1$ and $\phi_1(\xi) \equiv 0$ for $|\xi| \ge 1.1$. Denote $\phi_2(z) = |l+z| + |l-z| - 2$ and note that for $|z| \le \frac{2}{3}$ we have $\phi_2(z) = \sum_{i,j=1}^2 h_{ij}(z) z_i z_j$ for some $h_{ij} \in C^\infty$. By (3.4) it suffices to show $\|F\|_{L^1_x(\mathbb{R}^2)} \lesssim \lambda^{-2}\mu^2$ for $F(x) = \int_{\mathbb{R}^2} \phi_2(\lambda^{-1}\xi)\phi_1(\mu^{-1}\xi) e^{i\xi\cdot x} d\xi$. This follows from a change of variable $\mu^{-1}\xi \to \xi$ and integration by parts. For the third estimate we note $\Delta^{-1}\nabla T^{(2)}_{\lambda l} b_0 = \Delta^{-1}\nabla T^{(2)}_{\lambda l}(b_0 - \overline{b_0})$ and apply Lemma 3.2. □

Lemma 3.4. *Let* $\mathrm{supp}(\widehat{b_0}) \subset \{|k| \le \mu\}$, $\mu \le \frac{1}{2}\lambda$. *For* $m^{(a)}_\ell = m^{(a)}_{\ell,\lambda,\mu}$, $a = 1, 2, 3$, *defined by*

$$b_0 T^{(2)}_{\lambda l} b_0 = \frac{\mu^2}{\lambda^2} \sum_{\ell=1}^2 \partial_{x_\ell} T_{m^{(1)}_\ell}(b_0, b_0), \quad (T^{(1)}_{\lambda l} b_0)\partial_{x_1} b_0 = \frac{\mu^2}{\lambda} \sum_{\ell=1}^2 \partial_{x_\ell} T_{m^{(2)}_\ell}(b_0, b_0),$$

$$(T^{(1)}_{\lambda l} b_0)\partial_{x_2} b_0 = \frac{\mu^2}{\lambda} \sum_{\ell=1}^2 \partial_{x_\ell} T_{m^{(3)}_\ell}(b_0, b_0),$$

we have[4] $\|K^{(a)}_\ell\|_{L^1(\mathbb{R}^4)} := \|\mathcal{F}^{-1}(m^{(a)}_\ell)\|_{L^1(\mathbb{R}^4)} \lesssim 1$ *with the implicit constants independent of* λ *and* μ.

Proof. Observe that for $|z| \le \frac{2}{3}$, $\phi(z) = |l+z| - |l-z| - 2l\cdot z = \sum_{i,j,k=1}^2 h_{ijk}(z) z_i z_j z_k$ for some $h_{ijk} \in C^\infty$. Choose $\phi_1 \in C_c^\infty(\mathbb{R}^2)$ such that $\phi_1(\xi) \equiv 1$ for $|\xi| \le 1$ and $\phi_1(\xi) \equiv 0$ for $|\xi| \ge 1.1$. By using parity of ϕ, we have

$$\widehat{b_0 T^{(2)}_{\lambda l} b_0}(k) = \frac{i}{4}\lambda \sum_{k'\in\mathbb{Z}^2} (\phi(\lambda^{-1}k') - \phi(\lambda^{-1}(k'-k))) \widehat{b_0}(k')\widehat{b_0}(k-k')$$

$$= -\frac{ik}{4} \cdot \sum_{k'\in\mathbb{Z}^2} \int_0^1 (\nabla\phi)(\lambda^{-1}(k'-\theta k)) d\theta \phi_1(\mu^{-1}k')$$

[4] Here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on $\mathbb{R}^2 \times \mathbb{R}^2$. See (3.6).

$$\phi_1(\mu^{-1}(k-k'))\widehat{b_0}(k')\widehat{b_0}(k-k').$$

Note that $(\partial_\ell\phi)(\frac{k'-\theta k}{\lambda})\phi_1(\frac{k'}{\mu})\phi_1(\frac{k-k'}{\mu}) = \lambda^{-2}\sum_{1\le i,j\le 2}\tilde{h}_{\ell ij}(\frac{k'-\theta k}{\lambda})(k'-\theta k)_i(k'-\theta k)_j\phi_1(\frac{k'}{\mu})\phi_1(\frac{k-k'}{\mu})$ where $\tilde{h}_{\ell ij}\in C_c^\infty(\mathbb{R}^2)$. The result then follows from (3.6) by checking the L^1 bound of the kernel. The case for $T^{(1)}_{\lambda l}$ is similar. □

Proof of Proposition 3.1. Rewrite (2.3) as

$$\nabla q_{n+1} \overset{\circ}{\approx} \underbrace{\Lambda f_{n+1}\nabla^\perp f_{n+1}+\nabla q_n}_{\text{Mismatch error}} + \underbrace{\Lambda f_{n+1}\nabla^\perp f_{\le n}+\Lambda f_{\le n}\nabla^\perp f_{n+1}}_{\text{Transport error}} \underbrace{-\nu\nabla\Lambda^{\gamma-1}f_{n+1}}_{\text{Dissipation error}}$$
$$=:\nabla q_M+\nabla q_T+\nabla q_D.$$

Frequency localization of q_{n+1} can be easily deduced from q_M, q_T, and q_D which are defined below. For convenience, we shall write $a_{j,n+1}$ as a_j in the computation below.
`Mismatch error.` By (2.6), we can further decompose the mismatch error as

$$\nabla q_M \overset{\circ}{\approx} (\boxed{\text{main}}+\nabla q_n)+\boxed{\text{non-oscillatory error}}+\boxed{\text{oscillatory error}}$$
$$\overset{\circ}{\approx} \nabla q_{M1}+\nabla q_{M2}+\nabla q_{M3}.$$

We first estimate q_{M1}. To ease the notation we write $a_j^{\text{per}} = 2\sqrt{\frac{r_n}{5\lambda_{n+1}}}\sqrt{C_0+\mathcal{R}^o_j\frac{q_n}{r_n}}$ and $a_j = P_{\le\mu_{n+1}}a_j^{\text{per}}$. By using a fattened frequency projection $\tilde{P}_{\le\mu_{n+1}}$ which is frequency localized to $\{|k|\le 4\mu_{n+1}\}$ and noting that $\lambda_n \ll \mu_{n+1}$, we have

$$\begin{aligned}
&-\frac{1}{4}\cdot(5\lambda_{n+1})\cdot\sum_{j=1}^{2} l_j^\perp(l_j\cdot\nabla)a_j^2+\nabla q_n-\nabla q_{M1}\\
&=-\frac{5}{4}\lambda_{n+1}\sum_{j=1}^{2} l_j^\perp(l_j\cdot\nabla)\tilde{P}_{\le\mu_{n+1}}((P_{\le\mu_{n+1}}a_j^{\text{per}})^2)+\nabla q_n-\nabla q_{M1}\\
&=-\frac{5}{4}\lambda_{n+1}\sum_{j=1}^{2} l_j^\perp(l_j\cdot\nabla)\tilde{P}_{\le\mu_{n+1}}\left(-2a_j^{\text{per}}P_{>\mu_{n+1}}a_j^{\text{per}}+(P_{>\mu_{n+1}}a_j^{\text{per}})^2\right)-\nabla q_{M1} \overset{\circ}{\approx} 0.
\end{aligned}$$

Thus we can solve $q_{M1}\in C_0^\infty(\mathbb{T}^2)$ as

$$q_{M1} = -\frac{5}{4}\lambda_{n+1}\sum_{j=1}^{2}\Delta^{-1}\nabla \cdot\left(l_j^\perp(l_j\cdot\nabla)\tilde{P}_{\le\mu_{n+1}}\left(-2a_j^{\text{per}}P_{>\mu_{n+1}}a_j^{\text{per}}+(P_{>\mu_{n+1}}a_j^{\text{per}})^2\right)\right). \tag{3.7}$$

Note that q_{M1} is frequency localized to $\{|k|\le 4\mu_{n+1}\}$. By Lemma 3.2, we obtain

$$\|q_{M1}\|_X \lesssim \log\mu_{n+1}\cdot\lambda_{n+1}\sum_{j=1}^{2}\|a_j^{\text{per}}\|_\infty\|P_{>\mu_{n+1}}a_j^{\text{per}}\|_\infty \tag{3.8}$$

$$\lesssim (\log \mu_{n+1})\lambda_{n+1}\|a_j\|_\infty \cdot \frac{1}{\mu_{n+1}^2}\|\Delta a_j^{\mathrm{per}}\|_\infty \tag{3.9}$$

$$\lesssim \log \mu_{n+1} \cdot (\mu_{n+1}^{-1}\lambda_n)^2 r_n. \tag{3.10}$$

Note that both $\boxed{\text{non-oscillatory error}}$ and $\boxed{\text{oscillatory error}}$ have zero means, so we define

$$q_{M2} = \Delta^{-1}\nabla \cdot \boxed{\text{non-oscillatory error}}, \quad q_{M3} = \Delta^{-1}\nabla \cdot \boxed{\text{oscillatory error}}$$

in $C_0^\infty(\mathbb{T}^2)$. To estimate q_{M2}, we claim that

$$\|\Delta^{-1}\nabla \cdot ((T_{n+1,j}^{(1)}a_j)\nabla^\perp a_j)\|_X + \|\Delta^{-1}\nabla \cdot (5\lambda_{n+1}(T_{n+1,j}^{(2)}a_j)a_j l_j^\perp)\|_X \lesssim r_n\lambda_{n+1}^{-2}\mu_{n+1}^2 \log \mu_{n+1}.$$

Indeed, by Lemma 3.4, $(T_{n+1,j}^{(2)}a_j)a_j$ can be written as a divergence form and hence we can apply Lemma 3.2 to get

$$\|\Delta^{-1}\nabla \cdot (5\lambda_{n+1}(T_{n+1,j}^{(2)}a_j)a_j l_j^\perp)\|_X \lesssim (\log \mu_{n+1})\lambda_{n+1}\left(\frac{\mu_{n+1}}{\lambda_{n+1}}\right)^2 \frac{r_n}{\lambda_{n+1}} \lesssim r_n \left(\frac{\mu_{n+1}}{\lambda_{n+1}}\right)^2 \log \mu_{n+1}.$$

The other term can be estimated similarly. Then, it leads to

$$\|q_{M2}\|_X \lesssim r_n\lambda_{n+1}^{-2}\mu_{n+1}^2 \log \mu_{n+1}. \tag{3.11}$$

Next we estimate q_{M3}. Denote $T_{n+1,j}^{(i)} = T_{5\lambda_{n+1}l_j}^{(i)}$ for $i, j = 1, 2$. By Lemma 3.3, we have

$$\|T_{n+1,j}^{(1)}a_j\|_\infty \lesssim \lambda_{n+1}^{-1}\mu_{n+1}^2\sqrt{\frac{r_n}{\lambda_{n+1}}}, \quad \|T_{n+1,j}^{(2)}a_j\|_\infty \lesssim \lambda_{n+1}^{-2}\mu_{n+1}^3\sqrt{\frac{r_n}{\lambda_{n+1}}}. \tag{3.12}$$

Since all terms in (oscillatory error) have the frequency localized to $\sim \lambda_{n+1}$ provided that $48\lambda_n \le \lambda_{n+1}$, the estimate for q_{M3} easily follows from (3.12):

$$\begin{aligned}
\|\Delta^{-1}\nabla \cdot (osc1)\|_X &\lesssim \sum_{j=1}^2 \|\Delta^{-1}\nabla \cdot (l_j \cdot \nabla a_j + T_{n+1,j}^{(2)}a_j)(\lambda_{n+1}a_j l_j^\perp \cos(10\lambda_{n+1}l_j \cdot x) \\
&\quad + \nabla^\perp a_j \sin(10\lambda_{n+1}l_j \cdot x))\|_X \\
&\lesssim \sum_{j=1}^2 \lambda_{n+1}\|\Delta^{-1}\nabla \cdot \left(a_j l_j \cdot \nabla a_j l_j^\perp \cos(10\lambda_{n+1}l_j \cdot x)\right)\|_X \\
&\quad + \|\Delta^{-1}\nabla \cdot \left(\nabla a_j \cdot l_j \nabla^\perp a_j \sin(10\lambda_{n+1}l_j \cdot x)\right)\|_X \\
&\quad + \lambda_{n+1}\|\Delta^{-1}\nabla \cdot \left(a_j T_{n+1,j}^{(2)}a_j l_j^\perp \cos(10\lambda_{n+1}l_j \cdot x)\right)\|_X \\
&\quad + \|\Delta^{-1}\nabla \cdot \left(T_{n+1,j}^{(2)}a_j \nabla^\perp a_j \sin(10\lambda_{n+1}l_j \cdot x)\right)\|_X \\
&\lesssim \sum_{j=1}^2 \|\nabla a_j\|_\infty \|a_j\|_\infty + \lambda_{n+1}^{-1}\|\nabla a_j\|_\infty \|\nabla^\perp a_j\|_\infty \\
&\quad + \|T_{n+1,j}^{(2)}a_j\|_\infty \|a_j\|_\infty + \lambda_{n+1}^{-1}\|T_{n+1,j}^{(2)}a_j\|_\infty \|\nabla^\perp a_j\|_\infty \\
&\lesssim \left(\frac{\lambda_n}{\lambda_{n+1}}\right) r_n.
\end{aligned}$$

Here we use the frequency localization assumption of q_n (note that q_n is frequency localized to $\leq 12\lambda_n$) to derive $\|\nabla a_j\|_\infty \lesssim \lambda_n\sqrt{\frac{r_n}{\lambda_{n+1}}}$. Similarly, we obtain

$$\|\Delta^{-1}\nabla\cdot(osc2)\|_X \lesssim \sum_{j=1}^{2}\|T^{(1)}_{n+1,j}a_j\|_\infty(\|a_j\|_\infty + \lambda_{n+1}^{-1}\|\nabla^\perp a_j\|_\infty) \lesssim \left(\frac{\lambda_n}{\lambda_{n+1}}\right) r_n.$$

The estimates for (osc3)–(osc6) are similar (using $2/\sqrt{5} \leq |l_1 \pm l_2| \leq 4/\sqrt{5}$) and therefore

$$\|q_{M3}\|_X \lesssim \left(\frac{\lambda_n}{\lambda_{n+1}}\right) r_n. \tag{3.13}$$

Combining (3.8), (3.11), and (3.13) and using $b > 1$, $\beta < 1$, we can find $\Lambda_M = \Lambda_M(\beta, b)$ such that for any $\lambda_0 \geq \Lambda_M$, we get $q_M = q_{M1} + q_{M2} + q_{M3} \in C_0^\infty(\mathbb{T}^2)$ satisfying (see also Appendix 5)

$$\|q_M\|_X \leq \frac{1}{3} r_{n+1}.$$

`Transport error.` Define

$$q_T = \Delta^{-1}\nabla\cdot(\Lambda f_{n+1}\nabla^\perp f_{\leq n} + \Lambda f_{\leq n}\nabla^\perp f_{n+1}) \in C_0^\infty(\mathbb{T}^2).$$

Since $\Lambda f_{n+1}\nabla^\perp f_{\leq n} + \Lambda f_{\leq n}\nabla^\perp f_{n+1}$ is frequency-localized to $\sim \lambda_{n+1}$, using $\|f_{\leq n}\|_{C^\alpha} \leq 100$, we get

$$\|q_T\|_X \lesssim \|f_{n+1}\|_\infty(\|\nabla^\perp f_{\leq n}\|_\infty + \|\Lambda f_{\leq n}\|_\infty) \leq C_\alpha \lambda_n^{1-\alpha}\sqrt{\frac{r_n}{\lambda_{n+1}}} \leq \frac{1}{3} r_{n+1}$$

for some constant $C_\alpha > 0$. Note that the last inequality amounts to requiring

$$\lambda_n^{1-\alpha-\frac{1}{2}\beta-\frac{1}{2}b+b\beta} \ll 1. \tag{3.14}$$

With the choice of $\alpha = \frac{1}{2} + \frac{\beta}{2b} - (b-1)^3$, we have

$$1 - \alpha - \frac{1}{2}\beta - \frac{1}{2}b + b\beta = \frac{1}{2}(b-1)(3\beta - 1) - (b-1)^2\frac{\beta}{2b} + (b-1)^3 =: c_* < 0.$$

Indeed, since $\beta < \frac{1}{3}$, we have $c_* < 0$ for $b = 1+$. Then we find $\Lambda_T = \Lambda_T(\beta, b)$ so that $\Lambda_T^{c_*} \ll 1$.

`Dissipation error.` We define $q_D = -\nu\Lambda^{\gamma-1} f_{n+1} \in C_0^\infty(\mathbb{T}^2)$ which satisfies

$$\|q_D\|_X \leq C_2\lambda_{n+1}^{\gamma-1}\|f_{n+1}\|_\infty \leq 5C_2\lambda_{n+1}^{\gamma-1}\sqrt{\frac{r_n}{\lambda_{n+1}}} \leq \frac{1}{3} r_{n+1},$$

for some $C_2 = C_2(\nu, \gamma) > 0$. Since $\beta < 3 - 2\gamma$, we can find sufficiently small $b_0 = b_0(\nu, \gamma, \beta)$ such that for any $1 < b < b_0 + 1$ there exists $\Lambda_D = \Lambda_D(\nu, \gamma, \beta, b)$ which leads the last inequality for any $\lambda_0 \geq \Lambda_D$.

Collecting the estimates, we obtain $\|q_{n+1}\|_X \leq r_{n+1}$ if $\lambda_0 > \Lambda_0 = \max(\Lambda_M, \Lambda_T, \Lambda_D)$. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. With no loss we take $C_0 = 2$ in Proposition 2.3. Fix $\nu \geq 0$, $0 < \gamma < \frac{3}{2}$ and choose $0 < \beta < \min\left(\frac{1}{3}, 3 - 2\gamma\right)$ to satisfy $(1 + \beta)/2 > \alpha_*$. We then choose $b - 1 \in (0, b_0)$ so that $\alpha = \frac{1}{2} + \frac{\beta}{2b} - (b - 1)^3 \geq \alpha_*$, where b_0 is defined in Proposition 3.1. Lastly, choose λ_0 as in Proposition 3.1. Set parameters as in (1.4). If necessary, we adjust λ_0 to have $\sum_{m=0}^{\infty} \lambda_m^{\alpha - \frac{1}{2} - \frac{\beta}{2b}} \leq 1$. Take the base step $(f_{\leq 0}, q_0) = (0, 0)$. At n^{th}-step, assume that $(f_{\leq n}, q_n) \in C_0^\infty(\mathbb{T}^2) \times C_0^\infty(\mathbb{T}^2)$ satisfies

- $(f_{\leq n}, q_n)$ solves (2.1).
- $\operatorname{supp}(\widehat{f_{\leq n}}) \subset \{|k| \leq 6\lambda_n\}$, $\operatorname{supp}(\widehat{q_n}) \subset \{|k| \leq 12\lambda_n\}$ and $\|q_n\|_X \leq r_n$,

$$\|f_{\leq n}\|_{C^\alpha(\mathbb{T}^2)} \leq 50 \sum_{m=1}^{n} \lambda_m^\alpha \sqrt{\frac{r_{m-1}}{\lambda_m}} \leq 100 \sum_{m=0}^{n-1} \lambda_{m+1}^{\alpha - \frac{1}{2} - \frac{\beta}{2b}} \leq 100.$$

Then by Proposition 3.1 and (2.10), at $(n + 1)^{\text{th}}$step, we find f_{n+1} and $q_{n+1} \in C_0^\infty(\mathbb{T}^2)$ satisfying

- (f_{n+1}, q_{n+1}) solves (2.3).
- $\operatorname{supp}(\widehat{f_{\leq n+1}}) \subset \{|k| \leq 6\lambda_{n+1}\}$, $\|f_{n+1}\|_{C^\alpha(\mathbb{T}^2)} \leq 50\lambda_{n+1}^\alpha \sqrt{\frac{r_n}{\lambda_{n+1}}}$, $\operatorname{supp}(\widehat{q_{n+1}}) \subset \{|k| \leq 12\lambda_{n+1}\}$, and $\|q_{n+1}\|_X \leq r_{n+1}$.

Thus the induction step can be closed and it remains to show that $f_{\leq n}$ converges to the desired weak solution. We first check its regularity. Clearly

$$\|f_{\leq n'} - f_{\leq n}\|_{C^\alpha} \lesssim \sum_{m=n}^{n'-1} \lambda_{m+1}^{\alpha - \frac{1}{2} - \frac{\beta}{2b}}, \quad \forall n' \geq n.$$

Thus $f_{\leq n} \to f \in C^\alpha(\mathbb{T}^2) \subset C^{\alpha_*}(\mathbb{T}^2)$. Now denote $\theta_n = \Lambda f_{\leq n}$ and $\theta = \Lambda f$. Clearly

$$\langle \theta_n \Lambda^{-1} \nabla^\perp \theta_n - \nu \Lambda^{\gamma - 2} \nabla \theta_n - \nabla q_n, \nabla \psi \rangle = 0, \quad \forall \psi \in C^\infty(\mathbb{T}^2).$$

We then rewrite the above as

$$\frac{1}{2} \langle \Lambda^{-\frac{1}{2}} \theta_n, \Lambda^{\frac{1}{2}} [\mathcal{R}^\perp, \nabla \psi] \theta_n \rangle + \nu \langle \Lambda^{-\frac{1}{2}} \theta_n, \Lambda^{\gamma + \frac{1}{2}} \psi \rangle + \langle q_n, \Delta \psi \rangle = 0, \quad \forall \psi \in C^\infty(\mathbb{T}^2).$$

Since $\Lambda^{-\frac{1}{2}} \theta_n \to \Lambda^{-\frac{1}{2}} \theta$ strongly in L^∞, Proposition 5.1 implies that θ solves (SQG). □

Finally we remark that our solution $\theta = \Lambda f$ has an almost explicit form. By using (2.10), we have

$$f = \sum_{n=0}^{\infty} \sum_{j=1}^{2} 2 \sqrt{\frac{r_n}{5\lambda_{n+1}}} \left(P_{\leq \mu_{n+1}} \sqrt{C_0 + R_j^o \frac{q_n}{r_n}} \right) \cos(5\lambda_{n+1} l_j \cdot x).$$

The leading term is an almost explicit Fourier series (one can take C_0 large) and thus our solution is nontrivial.

5. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 based on the following proposition.

Proposition 5.1. *Let $\mathcal{R} = \mathcal{R}_j$, $j = 1, 2$. Assume $\phi \in H^3$ and $\theta \in \dot{H}^{-\frac{1}{2}}$ ($\overline{\theta} = 0$). Then we have*

$$\|[\mathcal{R}, \phi]\theta\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\phi\|_{\dot{H}^3} \|\theta\|_{\dot{H}^{-\frac{1}{2}}}.$$

Proof. Denote $m(k) = \frac{k_1}{|k|}$. It suffices to show that

$$\Big\| \sum_{k' \neq 0, k} |k|^{\frac{1}{2}} (m(k) - m(k')) \widehat{\phi}(k - k') \widehat{\theta}(k') \Big\|_{l_k^2} \lesssim \| |k|^3 \widehat{\phi}(k) \|_{l_k^2} \| |k|^{-\frac{1}{2}} \widehat{\theta}(k) \|_{l_k^2}. \tag{5.1}$$

If $|k'| \lesssim |k - k'|$, then $|k| \lesssim |k - k'|$, and

$$\text{LHS of}(5.1) \lesssim \Big\| \sum_{k' \neq 0, k} |k - k'| |\widehat{\phi}(k - k')| \cdot |k'|^{-\frac{1}{2}} |\widehat{\theta}(k')| \Big\|_{l_k^2} \lesssim \text{RHS of}(5.1).$$

If $|k - k'| \ll |k|$, then $|k| \sim |k'|$, and it suffices to use $|m(k) - m(k')| \lesssim |k - k'|(|k'| + |k|)^{-1}$. □

Proof of Theorem 1.4. The point is to use the weak formulation (below $\langle , \rangle$ denotes L^2-inner product in (t, x), and ψ is a time-dependent test function)

$$\langle \partial_t \theta_n, \psi \rangle + \frac{1}{2} \langle \Lambda^{-\frac{1}{2}} \theta_n, \Lambda^{\frac{1}{2}} [\mathcal{R}^{\perp}, \nabla \psi] \theta_n \rangle + \nu \langle \Lambda^{-\frac{1}{2}} \theta_n, \Lambda^{\gamma + \frac{1}{2}} \psi \rangle = 0.$$

By using the above together with Proposition 5.1, we have[5] $\|\partial_t \theta_n\|_{L_t^1 \dot{H}^{-8}} \lesssim 1$. Fix any $0 \neq k \in \mathbb{Z}^2$. We have $\|\partial_t \widehat{\theta_n}(k, t)\|_{L_t^1} \lesssim |k|^8$ and $\|\widehat{\theta_n}(k, t)\|_{L_t^2} \lesssim |k|^{-s}$. By further using a diagonal argument, we obtain along a subsequence

$$\|\widehat{\theta_{n_l}}(k, t) - \widehat{f}(k, t)\|_{L_t^2} \to 0 \quad \text{for any fixed } k. \tag{5.2}$$

Using $\sup_l \|\theta_{n_l}\|_{L_t^2 \dot{H}^s} \lesssim 1$ (note that $s > -\frac{1}{2}$), we have for any integer J (below $P_{>J}$ denotes frequency projection to the regime $|k| \geq 2^J$)

$$\|P_{>J}(\theta_{n_l} - f)\|_{L_t^2 \dot{H}^{-\frac{1}{2}}} \lesssim 2^{-J(s+\frac{1}{2})} \|\theta_{n_l} - f\|_{L_t^2 \dot{H}^s} \tag{5.3}$$

$$\lesssim 2^{-J(s+\frac{1}{2})}. \tag{5.4}$$

By (5.2) and (5.4), one obtains the strong convergence $\theta_{n_l} \to f$ in $L_t^2 \dot{H}^{-\frac{1}{2}}$. Since $\|\Lambda^{\frac{1}{2}} [\mathcal{R}^{\perp}, \nabla \psi](\theta_n - f)\|_2 \lesssim \|\theta_n - f\|_{\dot{H}^{-\frac{1}{2}}}$, it follows that f is the desired weak solution. □

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[5] Here t belongs to an arbitrary compact interval.

Appendix A: Bookkeeping of Various Parameters

In this appendix we sketch how the choice of various parameters in (1.4) take effect on various error terms and the regularity of the weak solution. Recall that (observe from below $\log \mu_{n+1} \sim \log \lambda_n$)

$$\lambda_n = \left\lceil \lambda_0^{b^n} \right\rceil, \quad r_n = \lambda_n^{-\beta}, \quad \mu_{n+1} = (\lambda_n \lambda_{n+1})^{\frac{1}{2}}, \quad \alpha = \frac{1}{2} + \frac{\beta}{2b} - (b-1)^3 > \frac{1}{2}.$$

Mismatch error $\quad r_n \frac{\lambda_n}{\lambda_{n+1}} \log \lambda_n \ll r_{n+1} \iff \lambda_n^{(b-1)(\beta-1)} \log \lambda_n \ll 1.$

Transport error $\quad \lambda_n^{1-\alpha} \sqrt{\frac{r_n}{\lambda_{n+1}}} \ll r_{n+1} \iff \lambda_n^{1-\alpha-\frac{1}{2}\beta-\frac{1}{2}b+b\beta} \ll 1.$

Dissipation error $\quad \lambda_{n+1}^{\gamma-1} \sqrt{\frac{r_n}{\lambda_{n+1}}} \ll r_{n+1} \iff \lambda_{n+1}^{\gamma-\frac{3}{2}+\beta-\frac{\beta}{2b}} \ll 1.$

C^α-regularity $\quad \lambda_{n+1}^{\alpha} \sqrt{\frac{r_n}{\lambda_{n+1}}} \ll 1 \iff \lambda_{n+1}^{\alpha-\frac{1}{2}-\frac{1}{2b}\beta} \ll 1.$

Now one can take $\alpha \approx \frac{1}{2} + \frac{\beta}{2b}$ to do a limiting computation. From the transport error we obtain (the limiting condition)

$$1 - \alpha - \frac{1}{2}\beta - \frac{1}{2}b + b\beta = \frac{1-b}{2b}(b - \beta(2b+1)) \Rightarrow \beta < \frac{1}{3}.$$

From the dissipation error we obtain $\frac{\beta}{2} < \frac{3}{2} - \gamma$.

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