Available online at www.sciencedirect.com
ScienceDirect
Journal of Differential Equations 478 (2026) 114622
Journal of Differential Equations
www.elsevier.com/locate/jde

Stability and convergence of an iterative low-regularity method for the Cahn-Hilliard equation

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Received 26 January 2026; revised 7 June 2026; accepted 25 June 2026

Abstract

Solutions of low regularity are important objects in the study of partial differential equations (PDEs). The Nobel laureate, physicist and chemist Onsager conjectured in 1949 [38] that loss of regularity may lead to “anomalous dissipation of energy” or “energy cascades” and cause turbulence in fluid dynamics. Therefore computing such solutions of low regularity is of much significance. In this paper we consider the low-regularity integrator method (LGI) to study the Cahn-Hilliard equations, avoiding the high regularity assumption while keeping the energy stability. In the literature a few works have already studied several LGI methods for various PDE models and proved the energy dissipation by either requiring a strong Lipschitz condition on the nonlinear source term or certain L^∞ bounds on the numerical solutions (maximum principle). However for models such as the (non-local) Cahn-Hilliard equation, the maximum principle no longer exists. As a result, solving such models via LGI-type method remains challenging for a long time. In this paper we aim to give a systematic approach on applying stabilized LGI-type methods to models where no maximum principle exists by solving the Cahn-Hilliard equation with stabilization and showing the energy dissipation. Our low-regularity integrator method requires no Lipschitz condition or L^∞ boundedness. Furthermore, we will show the L^2 convergence under minimal $u \in H^6$ regularity assumption (while keeping good stability) by a newly developed iteration method.

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https://doi.org/10.1016/j.jde.2026.114622

Keywords: Low regularity method; Partial differential equations; Energy stability; Fourier method

1. Introduction

1.1. Introduction to the models and historical review

Many mathematical models of physical and biological phenomena can be described via partial differential equations (PDEs), which are of essential importance in areas such as material sciences, biology, and kinetic theory. In this work, we focus on two classic PDE models: the Allen–Cahn equation (AC) and the Cahn–Hilliard equation (CH).

The Allen–Cahn model was first introduced in [2] to study the competition between crystal grain orientations in a binary alloy, while the Cahn–Hilliard equation was developed in [10] to model phase separation and phase transitions. Their precise formulations are given as follows:

$$\begin{cases} \partial_t u = \nu\Delta u - f(u), \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0, \end{cases} \tag{AC}$$

and

$$\begin{cases} \partial_t u = \Delta(-\nu\Delta u + f(u)), \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0. \end{cases} \tag{CH}$$

Here, the real-valued function $u(x,t)$ represents the mixture state of two phases, with $u = -1$ and $u = +1$ corresponding to the pure states of each phase. The parameter ν is small, and we occasionally denote $\varepsilon = \sqrt{\nu}$ to represent the transition layer width. Specifically, we take the spatial domain Ω to be the two-dimensional torus $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$, though our analysis can be extended to bounded domains with Dirichlet or Neumann boundary conditions. The nonlinear potential is typically chosen as

$$f(u) = F'(u) = u^3 - u, \quad F(u) = \frac{1}{4}(u^2-1)^2,$$

which leads to the energy functional

$$E(u) = \int_\Omega \left(\frac{1}{2}\nu|\nabla u|^2 + F(u)\right) \mathrm{d}x. \tag{1.1}$$

It is worth mentioning here that all results in the paper are for such form of $f(u)$ for the sake of simplicity, but can be extended to other double-well potential cases.

Both equations are gradient flows and dissipate the energy (1.1) over time. This property ensures an a priori H^1-norm bound for global well-posedness and serves as a key criterion for numerical stability. In the limit as $\varepsilon \to 0$, the Allen–Cahn equation converges to motion by mean curvature, while the Cahn–Hilliard equation converges to the Mullins–Sekerka flow. We refer to [30,39,1,26] for detailed asymptotic and rigorous analyses. For this reason, Allen-Cahn and

Cahn-Hilliard equations play important roles also in differential geometry (minimal surfaces etc.), we refer the readers to for [22–24] example.

The Cahn–Hilliard equation also plays a vital role in fluid dynamics, particularly when coupled with the Navier–Stokes equations to model multiphase flows in a thermodynamically consistent manner. Through a diffuse-interface approach, the Navier–Stokes–Cahn–Hilliard system (and related models such as the Navier–Stokes–Q-tensor system) captures complex interfacial phenomena—such as phase separation, droplet coalescence and breakup, and dynamic contact lines—without explicit interface tracking. On the other hand, studying solutions of low regularity is fundamental in the area of fluid PDE theory. The Nobel laureate Onsager conjectured in 1949 [38] that a loss of regularity could lead to anomalous energy dissipation or energy cascades, contributing to turbulence in fluid dynamics. Therefore, accurately computing such "rough" solutions is of considerable significance. This work aims to approach the limiting behaviors of (CH) in a suitable low-regularity framework.

Numerous numerical approaches have been developed for simulating the Cahn–Hilliard equation and related phase-field models. These methods can be broadly classified according to their treatment of time discretization and spatial approximation. In terms of time-stepping strategies, both fully explicit (e.g., forward Eule [4]) and semi-implicit (implicit-explicit) schemes [15,29,36,34,35] have been widely studied. To enable larger time steps while preserving structural properties, methods such as convex splitting [25] and stabilized semi-implicit schemes [29,42,48,49,35,36,34,16,17] have been introduced. For spatial discretization, common approaches include spectral methods [8,14,35,46,15,29,37], finite element methods [9,27,31], and finite difference methods [6,7]. We also list several results for other related gradient flow models here [11,20,28,21,40,44,45] and etc. These numerical approximations successfully capture both quantitative and qualitative features of the solutions, with a key focus being the preservation of energy dissipation.

A central issue in the numerical analysis of such gradient flows is energy stability. For instance, Feng and Prohl [27] derived error estimates for a finite element discretization of the Cahn–Hilliard equation, though the bounds depend strongly on the interface parameter ν. While fully explicit methods typically suffer from severe time-step constraints and generally fail to guarantee energy decay, semi-implicit methods—where the linear part is treated implicitly and the nonlinearity explicitly—offer a more efficient alternative. However, standard semi-implicit schemes can still become unstable for large time steps. To address this, stabilized versions have been developed that incorporate auxiliary stabilizing terms to ensure energy decay under milder conditions, cf. [29,48,49,43]. Nevertheless, many of these methods rely on strong Lipschitz assumptions or a priori L^∞-bounds on the numerical solution. Recent advances [34,35,16] have employed sharper analytical tools developed in [5,33] to relax these requirements.

Inspired by progress in semi-implicit schemes, so-called low-regularity integrators (LGI) have emerged as an effective approach for simulating rough solutions, particularly in the context of equations with limited regularity [32,3,12,18,41,47]. These methods aim to accurately capture energy dissipation and other structural properties even under low-regularity conditions. In a short word, the LGI methods discretize the semi-group presentation of the solutions to nonlinear PDEs. In more details, take the Allen-Cahn equation as an example. It is well-known that the smooth solution to (AC) is given by the semi-group mild form (or Duhamel's Formula):

$$u(t) = e^{\nu t\Delta}u(t_0) + \int_{t_0}^{t} e^{\nu(t-\sigma)\Delta} f(u(\sigma))\ \mathrm{d}\sigma, \tag{1.2}$$

where $e^{t\Delta}$ is the standard heat semi-group (heat kernel). Then the idea of the classic integrator methods is to discretize the semi-group presentation in each time interval (t_n, t_{n+1}) by approximating the integrand function by a constant (cf. [32]):

$$\begin{aligned} u(t_{n+1}) =& e^{\nu\tau\Delta} u(t_n) + \int_{t_n}^{t_{n+1}} e^{\nu(t_{n+1}-\sigma)\Delta} f(u(\sigma))\, \mathrm{d}\sigma \\ \approx& e^{\nu\tau\Delta} u(t_n) + \int_{t_n}^{t_{n+1}} e^{\nu(t_{n+1}-\sigma)\Delta} f(u(t_n))\, \mathrm{d}\sigma \\ =& e^{\nu\tau\Delta} u(t_n) + \frac{1 - e^{\nu(t_{n+1}-t_n)\Delta}}{-\nu\Delta} f(u(t_n)), \end{aligned} \tag{1.3}$$

or equivalently approximating $u(t_n) \approx u^n$, $u(t_{n+1}) \approx u^{n+1}$ and $\tau = t_{n+1} - t_n$ we have the following LGI method for the usual Allen-Cahn equation:

$$u^{n+1} = e^{\nu\tau\Delta} u^n + \frac{e^{\nu\tau\Delta} - 1}{\nu\Delta} f(u^n). \tag{1.4}$$

One of main advantages of LGI method (1.4) is the lower regularity assumption of the exact solution. To see this we first recall the usual semi-implicit time stepping method:

$$\frac{u^{n+1} - u^n}{\tau} = \nu\Delta u^n - f(u^n). \tag{1.5}$$

It is worth remarking here that it is well known the truncation error of the semi-implicit time stepping method (1.5) is of order $O(\tau \cdot \partial_{tt} u)$; therefore the minimal regularity requirement to guarantee the convergence is $\partial_{tt} u \approx \Delta^2 u \in L^2$, or $u \in H^4$. We then take a brief look of the truncation error of LGI method (1.4). Clearly from (1.3), we can derive that

$$u(t_{n+1}) = e^{\nu\tau\Delta} u(t_n) + \frac{e^{\nu(t_{n+1}-t_n)\Delta} - 1}{\nu\Delta} f(u(t_n)) + R_n, \tag{1.6}$$

where

$$R_n = \int_{t_n}^{t_{n+1}} e^{\nu(t_{n+1}-\sigma)\Delta} \left(f(u(\sigma)) - f(u(t_n)) \right) \mathrm{d}\sigma\,. \tag{1.7}$$

Noting that by the maximum principle of Allen-Cahn equation (cf. [16]) we can estimate R_n as

$$\|R_n\|_2 \lesssim \tau \max_{\sigma \in [t_n, t_{n+1}]} \|u(\sigma) - u(t_n)\|_2, \tag{1.8}$$

where the $\|\cdot\|_2$ is the L^2-norm that will be defined in Section 2. Recall that from the continuous equation indeed one has

$$u(\sigma) - u(t_n) = \nu \int_{t_n}^{\sigma} \Delta u(\tilde{\sigma}) \, \mathrm{d}\tilde{\sigma} - \int_{t_n}^{\sigma} f(u(\tilde{\sigma})) \, \mathrm{d}\tilde{\sigma} \, . \tag{1.9}$$

As a result, the error $\|R_n\|_2 \lesssim \tau^2 \max_{\sigma\in[0,T]} \|\Delta u(\sigma)\|_2$. It is clear that the regularity assumption is therefore $\Delta u \in L^2$ or $u \in H^2$. Compared to usual semi-implicit time stepping, this LGI method requires much lower regularity. One can easily check the semi-implicit method requires $u \in H^8$ in the Cahn-Hilliard case where the LGI method requires $u \in H^6$.

However, the idea above either requires a strong Lipschitz condition on the nonlinear source term, or requires certain L^∞ bounds on the numerical solutions. Such requirement is satisfied by the Allen-Cahn equation by the standard maximum principle (we refer the readers to [34,16] for example). While for other models such as the (non-local) Cahn-Hilliard equation, the maximum principle no longer exists. As a result, solving PDEs via LGI method where strong Lipschitz conditions or maximum principle are no longer applicable remains open for a long time. One of our modest goals of this paper is to give a systematic approach on applying LGI-type methods to such models. Another major difficulty arises from proving the L^2 convergence under the minimal regularity assumption. Indeed by directly applying the usual discrete Grönwall's inequalities one might need a much higher regularity assumption to obtain the desired $O(\tau)$-error. Therefore we aim to show the L^2 convergence under minimal regularity assumption while keeping good stability.

To be more specific, we consider the following stabilized LGI method for the Cahn-Hilliard equation (CH):

$$\begin{cases} u^{n+1} = e^{-\nu\Delta^2\tau} u^n + \dfrac{1 - e^{-\nu\Delta^2\tau}}{\nu\Delta} \Pi_N[f(u^n)] - S\tau\Delta^2(u^{n+1} - u^n), \\ u^0 = \Pi_N u_0, \end{cases} \tag{1.10}$$

where τ is the time step, $S > 0$ is the coefficient for the $O(\tau)$ regularization term and Π_N is truncation of Fourier modes of L^2 functions to $|k|_\infty \le N$, where $k = (k_1, k_2) \in \mathbb{Z}^2$, $|k|_\infty = \max\{|k_1|, |k_2|\}$. In fact (1.10) above can be reorganized and rewritten into the following form:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} + S\Delta^2(u^{n+1} - u^n) = \dfrac{e^{-\nu\Delta^2\tau} - 1}{\tau} u^n + \dfrac{1 - e^{-\nu\Delta^2\tau}}{\tau\nu\Delta} \Pi_N[f(u^n)], \\ u^0 = \Pi_N u_0. \end{cases} \tag{1.11}$$

Remark 1.1. It is worth mentioning that from (1.11) one can observe that our integrator method behaves very similarly to an explicit forward Euler method since

$$\begin{aligned} \frac{e^{-\nu\Delta^2\tau} - 1}{\tau} &= -\nu\Delta^2 + \frac{1}{2}\nu^2\Delta^4\tau + O(\tau^2), \\ \frac{1 - e^{-\nu\Delta^2\tau}}{\tau\nu\Delta} &= \Delta - \frac{1}{2}\nu\Delta^3\tau + O(\tau^2). \end{aligned}$$

Therefore informally speaking, (1.11) can be understood as the usual explicit forward Euler method for (CH) with a stabilization term:

$$\frac{u^{n+1}-u^n}{\tau}+S\Delta^2(u^{n+1}-u^n)=-\nu\Delta^2u^n+\Delta\Pi_N[f(u^n)]+O(\nu\tau,\tau^2). \tag{1.12}$$

Remark 1.2. The stabilizer $S\Delta^2(u^{n+1}-u^n)$ is not unique. In fact our analysis works for many other stabilizers such as $Se^{-\Delta}(u^{n+1}-u^n)$ and so on. Another particular choice is $\frac{\nu}{2}\Delta^2(u^{n+1}-u^n)-S\Delta(u^{n+1}-u^n)$. Such particular choice results in the analysis of the energy dissipation. More precisely speaking, we need $\frac{\nu}{2}\Delta^2(u^{n+1}-u^n)$ to balance the drawback caused by the "explicit forward" diffusion; the real stabilizer is indeed $-S\Delta(u^{n+1}-u^n)$. We choose $S\Delta^2(u^{n+1}-u^n)$ just for the sake of simplicity. However due to such choice, the regularity assumption will be higher and we refer to more discussion in Remark 1.11. The stabilizers discussed above are of order $O(\tau)$, indeed stabilizers of order $O(\tau^2)$ such as $\tau S\Delta^2(u^{n+1}-u^n)$ can be introduced; however the energy stability will be weaker in the way that the time step τ needs to be smaller (than certain quantity of ν). Even though such choices of stabilizers are not unique, they are necessary. As will be presented in Section 5 with numerical evidence supporting, we will see examples without stabilizers have growing energy.

Remark 1.3. The exact explicit forward Euler method seems challenging to obtain the energy stability because it seems unlikely to obtain a uniform Sobolev H^s-bound estimate. To our best knowledge, we believe such problem arises from the essence of the forward Euler method since the regularity at step n is higher than the regularity at step $n+1$ due to the "explicit forward" diffusion $-\Delta^2u^n$. This seems to be the crucial technical difference between our integrator method (1.11) and the exact explicit forward Euler method (1.12) without the stabilizer $S\Delta^2(u^{n+1}-u^n)$ and higher order $O(\nu\tau,\tau^2)$ perturbation.

Remark 1.4. As mentioned earlier in Remark 1.1, our method (1.11) can be understood as an explicit forward Euler method. It is well known that after spatial discretization on a grid of size h, any explicit method will have a time step τ restriction of size $\tau<O(h^4)$ for the fourth order (CH). However as will be explained later (see Theorem 1.8), in order to show the convergence of our numerical scheme, we only require $\tau<(\log(N))^{-1}$, which is a logarithmically mild restriction. Moreover the energy stability holds for any size of τ, see Theorem 1.6.

Remark 1.5. It is worth emphasizing here that Cahn-Hilliard equation indeed shares the mass conservation: $\frac{d}{dt}\int_{\mathbb{T}^2}u\,dx=0$ resulting from the integration by parts. Therefore we can assume $\int_{\mathbb{T}^2}u=0$ or $\widehat{u}(0)=0$ without loss of generality. Here $\widehat{u}(k)$ is the Fourier series of u, see Section 2 for the notation.

1.2. Main results

Our main results state below.

Theorem 1.6 *(Unconditional energy stability for* (CH)*). Consider* (1.10) *or (1.11) with $\nu>0$ and assume $u_0\in H^2(\mathbb{T}^2)$. Then there exists a constant β_0 depending only on the initial energy $E_0=E(u_0)$ such that if*

$$S\geq\beta\cdot\left(\|u_0\|_{H^2}^2+\nu^{-1}|\log\nu|+\nu^{-1}\log N+\nu\right),\quad \beta\geq\beta_0.$$

Then $E(u^{n+1}) \le E(u^n)$, $\forall n \ge 0$ and for any choice of the time step τ, where the energy E is defined in (1.1).

Remark 1.7. The choice of S is not optimal here; in fact we refer the readers to Section 5 for numerical evidence with much smaller S. Determining the optimal bound of the stabilizers can be a very interesting question, however it is not the focus of the presenting paper; we leave this question to the readers.

Theorem 1.8 *(L^2 error estimate). Fix $\nu > 0$ and let $u_0 \in H^s$, $s \ge 6$. Let $0 < \tau \le D$ for some $D > 0$. Let $u(t)$ be the continuous solution to the 2D Cahn-Hilliard equation* (CH) *with initial data u_0. Let u^m ($1 \le m \le M$, where M is a fixed iteration number) be defined in (1.11) with initial data u^0. Assume S satisfies the same condition in Theorem 1.6. Define $t_0 = 0$ and $t_m = m\tau$ for $m \ge 1$. Then for any m that satisfies $1 \le m \le M$,*

$$\|u^m - u(t_m)\|_2 \le (1+S) \cdot C_2 \cdot e^{C_1 t_m} \left(N^{-s} + \tau + \tau^{\frac{s}{8}} + \tau \cdot N^{-s+4} \right),$$

where $C_1 > 0$ is a constant depending on ν , u_0; $C_2 > 0$ is a constant depending on s , ν and u_0.

Remark 1.9. It is worth mentioning that the requirement $S \ge \beta \nu^{-1} \log N$ in Theorem 1.6 is troublesome in the convergence of the numerical scheme. However as stated in Theorem 1.8 above, $S \cdot N^{-s} \to 0$ as $N \to \infty$ therefore such choice of S will not ruin the convergence. On the other hand, in order the numerical scheme to converge we do need $(\tau + \tau^{\frac{s}{8}}) \cdot S \to 0$ as $\tau \to 0$, which requires $\tau < (\log N)^{-1}$. This is still a very mild restriction as discussed in Remark 1.4 because if one uses spatial discretization, the restriction on the size of the time step τ is usually much smaller (of polynomial order of $\frac{1}{N}$) to be numerically stable.

Remark 1.10. Here we need to require that τ is not arbitrarily large. However, in practice it is not a big issue as we always use small (or at least not arbitrarily large) time steps.

Remark 1.11. Here the choice of $s \ge 6$ is mainly due to the stabilization term of high regularity; indeed if worse stability is allowed (much smaller time step τ) then we can derive the L^2 error under even weaker assumption than the indicated one from the truncation error analysis. On the other hand, the convergence rate $\tau^{\frac{s}{8}}$ is technical. In fact if we allow $s \ge 8$, as the usual semi-implicit time stepping does, then the desired 1st order convergence can be recurred. It is also worth pointing out that there are already works such as [32] that have developed strategies for systems of less regular initial data such as incompressible Navier-Stokes system. In fact due to the incompressibility of Navier-Stokes equation the energy dissipation is natural and no longer challenging.

Remark 1.12. The major difficulty of proving Theorem 1.8 arises from the minimal H^6 regularity assumption. Indeed by directly applying the usual discrete Grönwall's inequalities we might need an H^{10} assumption to obtain the desired $O(\tau)$-error. Such phenomenon is mainly due to the lack of control of $L^2 G^n$. To overcome this difficulty we developed a novel iterative discrete Grönwall framework and successfully lower the requirement to H^6. For more discussion we refer the readers to Section 5.

1.3. Organization of the presenting paper

The presenting paper is organized as follows. In Section 2 we list the notation and preliminaries including several useful lemmas. The long-term asymptotic behaviors and the smoothing effect of the Cahn-Hilliard equation are shown in Section 3. The energy stability of the LGI method of the 2D Cahn-Hilliard will be shown in Section 4 while the error estimate is given in Section 5. Numerical experiments will be presented in Appendix Section A. Further discussion and concluding remarks will be in Section 6.

2. Notation and preliminaries

Throughout this paper, for any two (non-negative in particular) quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We shall denote $X \lesssim_{Z_1, Z_2, \cdots, Z_k} Y$ if $X \le CY$ and the constant C depends on the quantities $Z_1, \cdots, Z_k$.

For a real-valued function $u : \Omega \to \mathbb{R}$ we denote its usual Lebesgue L^p-norm by

$$\|u\|_p = \|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \text{ess sup}_{x \in \Omega} |u(x)|, & p = \infty. \end{cases} \tag{2.1}$$

Suppose $u \in L^p(\Omega)$ and all weak derivatives $\partial^\alpha u$ exist for $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$, such that $\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, then we denote $u \in W^{k,p}(\Omega)$ to be the standard Sobolev space. The corresponding norm of $W^{k,p}(\Omega)$ states below:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p \, dx \right)^{\frac{1}{p}} .$$

For $p = 2$ case, we use the convention $H^k(\Omega)$ to denote the space $W^{k,2}(\Omega)$. We often use $D^m u$ to denote any differential operator $D^\alpha u$ for any $|\alpha| = m$.

In this paper we use the following convention for Fourier expansion on $\mathbb{T}^d$:

$$f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{ik\cdot x} \ , \ \widehat{f}(k) = \int_\Omega f(x) e^{-ik\cdot x} \, dx \ .$$

Taking advantage of the Fourier expansion, we use the well-known equivalent H^s-norm and $\dot{H}^s$-semi-norm of function f by

$$\|f\|_{H^s} = \frac{1}{(2\pi)^{d/2}} \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^{2s}) |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} \ , \ \|f\|_{\dot{H}^s} = \frac{1}{(2\pi)^{d/2}} \left(\sum_{k \in \mathbb{Z}^d} |k|^{2s} |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} .$$

In addition, for $N \ge 2$, we define

$$X_N = \text{span} \left\{ \cos(k \cdot x) \ , \sin(k \cdot x) : \ k = (k_1, k_2) \in \mathbb{Z}^2 \ , \ |k|_\infty = \max\{|k_1|, |k_2|\} \le N \right\} .$$

Lemma 2.1 *(Sobolev inequality on $\mathbb{T}^d$). Let $0 < s < d$ and $f \in L^q(\mathbb{T}^d)$ for any $\frac{d}{d-s} < p < \infty$, then*

$$\| \langle \nabla \rangle^{-s} f \|_{L^p(\mathbb{T}^d)} \lesssim_{s,p,d} \| f \|_{L^q(\mathbb{T}^d)} , \text{ where } \frac{1}{q} = \frac{1}{p} + \frac{s}{d} ,$$

where $\langle \nabla \rangle^{-s}$ denotes $(1-\Delta)^{-\frac{s}{2}}$ and $A \lesssim_{s,p,d} B$ is defined as $A \le C_{s,p,d} B$ where $C_{s,p,d}$ is a constant dependent on s, p and d.

Remark 2.2. Note that this Sobolev inequality is a variety of the standard version. Note that on the Fourier side the symbol of $\langle \nabla \rangle^{-s}$ is given by $(1+|k|^2)^{-\frac{s}{2}}$. In particular, $\| f \|_{\infty(\mathbb{T}^d)} \lesssim \| f \|_{H^2(\mathbb{T}^d)}$, is known as Morrey's inequality. Moreover we define the symbol $|\nabla| f$ (or alternatively $|\nabla|^{-1} f$) given by $|k| \widehat{f}(k)$ (or alternatively $|k|^{-1} \widehat{f}(k)$) from the Fourier side.

Remark 2.3. If one further requires that $f \in H^1(\mathbb{T}^2)$ has zero mean, i.e. $\widehat{f}(0) = 0$ we have

$$\| f \|_4 \le \| f \|_2^{\frac{1}{2}} \| \nabla f \|_2^{\frac{1}{2}}, \tag{2.2}$$

and

$$\| f \|_6 \le \| f \|_2^{\frac{1}{3}} \| \nabla f \|_2^{\frac{2}{3}}. \tag{2.3}$$

The proof of (2.2) and (2.3) follows from Lemma 2.1 and standard interpolation.

Lemma 2.4. *Suppose $f \in H^1(\mathbb{T}^2)$ and f has zero mean, i.e. $\widehat{f}(0) = 0$. Then*

$$\| f \|_2 \le \| |\nabla|^{-1} f \|_2^{\frac{1}{2}} \| \nabla f \|_2^{\frac{1}{2}}, \tag{2.4}$$

$$\| f \|_2 \le \| \nabla f \|_2. \tag{2.5}$$

Proof. The proof follows from Parseval's identity directly:

$$\int_{\mathbb{T}^2} f^2 \, dx = \int_{\mathbb{T}^2} |\nabla|^{-1} f \cdot |\nabla| f \, dx.$$

Then (2.4) follows from the classic Cauchy-Schwarz inequality by noting $\| |\nabla| f \|_2 = \| \nabla f \|_2$. (2.5) follows from the standard Poincaré inequality (or directly from the Fourier side). □

Lemma 2.5 *(Log-type interpolation). For all $f \in H^s(\mathbb{T}^2)$, $s > 1$ and suppose f has zero mean, i.e. $\widehat{f}(0) = 0$, then*

$$\| f \|_\infty \le C_s \cdot \left(\| f \|_{\dot{H}^1} \sqrt{\log(\| f \|_{\dot{H}^s} + 3)} + 1 \right) . \tag{2.6}$$

Here C_s is a constant which only depends on s.

Proof. This lemma is a special case of Lemma 3.1 in [16] and we refer the readers to the proof therein. □

Lemma 2.6 *(Discrete Grönwall's inequality). Let $\tau > 0$ and $y_n \geq 0$, $\alpha_n \geq 0$, $\beta_n \geq 0$ for $n = 1, 2, 3 \cdots$. Suppose*

$$\frac{y_{n+1} - y_n}{\tau} \leq \alpha_n y_n + \beta_n \ , \ \forall\, n \geq 0 \, .$$

Then for any $m \geq 1$, we have

$$y_m \leq \exp\left(\tau \sum_{n=0}^{m-1} \alpha_n \right) \left(y_0 + \tau \sum_{k=0}^{m-1} \beta_k \right) \, .$$

Proof. The proof of this lemma is standard, cf. [13]. □

3. Long-time asymptotics, smoothing effect and temporal integrability

Assume that $u(x,t)$ is a smooth solution to the Cahn-Hilliard equations, it is clear that the energy dissipation provides an *a priori* H^1-norm bound and since the scaling-critical space for (CH) is L^2 in 2D (and $\dot{H}^{\frac{1}{2}}$ in 3D), the global well-posedness follow from standard energy estimates. Therefore we first show the following H^k boundedness of the Cahn-Hilliard equations.

Lemma 3.1 *(H^k boundedness). Assume $u(x,t)$ is a smooth solution to the Cahn-Hilliard equation in $\mathbb{T}^2$ and the initial data $u_0 \in H^k(\mathbb{T}^2)$ for $k \geq 2$. Then,*

$$\sup_{t \geq 0} \|u(t)\|_{H^k(\mathbb{T}^2)} \lesssim_k 1 \tag{3.1}$$

where we omit the dependence on ν and u_0.

Proof of Lemma 3.1. The proof is based on the kernel estimate and we sketch the details. To start with we can write the solution u in the mild form

$$u(t) = e^{-\nu t \Delta^2} u_0 + \int_0^t e^{-\nu(t-\sigma)\Delta^2} \Delta(u^3 - u) \ \mathrm{d}\sigma \ .$$

Note that by the energy dissipation we indeed have $\|u\|_{H^1(\mathbb{T}^2)} \lesssim 1$ for any $t > 0$. Therefore it suffices to this argument inductively, namely we show $\|u\|_{H^2} \lesssim 1$ for any $t \geq 0$. Then by taking any second order spatial derivative and L^2 norm in the formula above, we derive

$$\|D^2 u\|_2 \leq \|D^2 e^{-\nu t \Delta^2} u_0\|_2 + \int_0^t \|D^2 e^{-\nu(t-\sigma)\Delta^2} \Delta(u^3 - u)\|_2 \ \mathrm{d}\sigma,$$

where $D^m u$ denotes any differential operator $D^\alpha u$ for any $|\alpha| = m$.

Firstly, we consider the nonlinear part. We indeed observe that

$$\|D^2 e^{-\nu(t-\sigma)\Delta^2} \Delta(u^3 - u)\|_2 \lesssim \|K_1 * (u^3 - u)\|_2,$$

where K_1 is the kernel corresponding to $\Delta D^2 e^{-\nu(t-\sigma)\Delta^2}$. (It is easy to see that here D can interchange with Δ.) Let $\gamma = t - \sigma$, it then suffices to estimate the following quantity:

$$\int_0^t \|K_1 * (u^3 - u)\|_2 \, \mathrm{d}\gamma = \int_0^1 \|K_1 * (u^3 - u)\|_2 \, \mathrm{d}\gamma + \int_1^t \|K_1 * (u^3 - u)\|_2 \, \mathrm{d}\gamma \,. \tag{3.2}$$

Here we assume $t > 1$ with no loss since the case $t < 1$ follows easily as a special case. For the region when $\gamma \geq 1$ we get

$$\|K_1 * (u^3 - u)\|_2 \lesssim \|K_1 * (u^3 - u)\|_\infty \lesssim \|K_1\|_2 \cdot \|u\|_6^3 \lesssim \|K_1\|_2,$$

by the standard Sobolev embedding $\|u\|_6 \lesssim \|u\|_{H^1} \lesssim 1$. Note that for $\gamma > 1$ we have

$$\begin{aligned}\|K_1\|_2 &\lesssim \left(\sum_{|k|\geq 1} |k|^8 e^{-2\nu\gamma|k|^4} \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_1^\infty e^{-2\nu\gamma r^4} r^9 \, \mathrm{dr} \right)^{\frac{1}{2}} \\ &\lesssim \gamma^{-\frac{1}{2}} e^{-\nu\gamma}.\end{aligned} \tag{3.3}$$

As a result, we have

$$\int_1^t \|K_1 * (u^3 - u)\|_2 \, \mathrm{d}\gamma \lesssim \int_1^t \gamma^{-\frac{1}{2}} e^{-\nu\gamma} \, \mathrm{d}\gamma \lesssim 1. \tag{3.4}$$

The arguments above indeed indicate a smoothing effect. The region $0 < \gamma < 1$ follows from the standard local theory.

We then can conclude that

$$\int_0^t \|D^2 e^{-\nu(t-\sigma)\Delta^2} \Delta(u^3 - u)\|_2 \, \mathrm{d}\sigma \lesssim 1;$$

the linear part follows from similar arguments. □

We then show the following smoothing effect and temporal integrability of the Cahn-Hilliard equations.

Lemma 3.2 *(Smoothing effect). Assume $u(x,t)$ is a smooth solution to the Cahn-Hilliard equation in $\mathbb{T}^2$ with the initial data $u_0 \in L^2(\mathbb{T}^2)$. Then it follows for any $k \geq 1$ that*

$$\sup_{t\geq 1} \|u(t)\|_{H^k(\mathbb{T}^2)} \leq C \tag{3.5}$$

where the dependence is merely on k, ν *and* u_0.

Proof. The proof is very similar to the proof of Lemma 3.1. To start with we again write the solution u in the mild form

$$u(t) = e^{-\nu t\Delta^2} u_0 + \int_0^t e^{-\nu(t-\sigma)\Delta^2} \Delta(u^3 - u)\, \mathrm{d}\sigma\,.$$

It suffices to prove this argument inductively, namely we will show $\|u\|_{H^1} \lesssim 1$ for any $t \geq 1$. Then by taking any first order spatial derivative and L^2 norm in the formula above, we derive that

$$\|Du\|_2 \leq \|De^{-\nu t\Delta^2} u_0\|_2 + \int_0^t \|e^{-\nu(t-\sigma)\Delta^2} D\Delta(u^3 - u)\|_2 \,\mathrm{d}\sigma,$$

where $D^m u$ denotes any differential operator $D^\alpha u$ for any $|\alpha| = m$.

`Linear part.` It is clear from the Fourier side that

$$\|De^{-\nu t\Delta^2} u_0\|_2^2 \lesssim \sum_{|k|\geq 1} e^{-2\nu t|k|^4} |k|^2 |\widehat{u_0}(k)|^2. \tag{3.6}$$

Note that $e^{-2\nu t|k|^4}|k|^2 \lesssim \frac{1}{\sqrt{t}}$ (for the details we refer the readers to the Appendix in [16]), then $t \geq 1$ implies that $e^{-2\nu t|k|^4}|k|^2 \lesssim 1$. As a result,

$$\|De^{-\nu t\Delta^2} u_0\|_2^2 \lesssim \|u_0\|_2^2. \tag{3.7}$$

`Nonlinear part.` Since $t > 1$, we will estimate as follows:

$$\begin{aligned} \int_0^t \|e^{-\nu(t-\sigma)\Delta^2} D\Delta(u^3 - u)\|_2 \,\mathrm{d}\sigma &= \int_0^t \|e^{-\nu\tau\Delta^2} D\Delta(u^3 - u)\|_2 \,\mathrm{d}\tau \\ = \int_0^1 \|K_2 * (u^3 - u)\|_2 \,\mathrm{d}\tau &+ \int_1^t \|K_2 * (u^3 - u)\|_2 \,\mathrm{d}\tau, \end{aligned} \tag{3.8}$$

where K_2 is the kernel corresponding to $\Delta D e^{-\nu\tau\Delta^2}$.

When $\tau \in [0, 1]$ we have

$$\|K_2 * (u^3 - u)\|_2^2 \lesssim \sum_{|k|\geq 1} e^{-2\nu\tau|k|^4} |k|^6 |\widehat{(u^3 - u)}(k)|^2. \tag{3.9}$$

Defining $g(x) = x^6 e^{-2\nu\tau x^4}$ one can check that $\max g(x) \lesssim \tau^{-\frac{3}{2}}$. Therefore $\|K_2 * (u^3 - u)\|_2 \lesssim \tau^{-\frac{3}{4}}$ and

$$\int_0^1 \|K_2 * (u^3 - u)\|_2 \, \mathrm{d}\tau \lesssim \int_0^1 \tau^{-\frac{3}{4}} \, \mathrm{d}\tau \lesssim 1.$$

Now for $\tau \in [1, t]$ we have

$$\int_1^t \|K_2 * (u^3 - u)\|_2 \, \mathrm{d}\tau \lesssim \int_1^t \|K_2 * (u^3 - u)\|_\infty \, \mathrm{d}\tau \lesssim \int_1^t \|K_2\|_2 \cdot \|u\|_6^3 \, \mathrm{d}\tau \, .$$

Similar to (3.3) we can obtain that for $\tau > 1$ that

$$\|K_2\|_2 \lesssim \tau^{-\frac{1}{2}} e^{-\nu\tau},$$

which shows that $\int_1^t \|K_2\|_2 \cdot \|u\|_6^3 \, \mathrm{d}\tau \lesssim \int_1^\infty \|K_2\|_2 \, \mathrm{d}\tau \lesssim 1$. Combining the estimate in linear and nonlinear parts we conclude that $\sup_{t\ge 1} \|u\|_{H^1} \lesssim 1$. Inductively we can conclude that $\sup_{t\ge 1} \|u\|_{H^k} \lesssim 1$ for any $k \ge 1$. □

Proposition 3.3 *(Temporal integrability). Let $s \ge 4$. Assume that $u(t,x)$ is the unique solution to the Cahn-Hilliard equations (CH) with initial data $u_0 \in H^s(\mathbb{T}^2)$ and of zero mean. Then we have for any $T \ge 0$ that following inequality holds:*

$$\int_0^T \|\partial_t \Delta u\|_2^2 \, \mathrm{dt} \le C(1 + T),$$

where $C > 0$ depends only on ν and u_0.

Proof. Firstly we assume $T \le 1$. Note that after applying Δ to the (CH) we can derive that

$$\partial_t \Delta u = -\nu \Delta^3 u + \Delta^2 (u^3 - u). \tag{3.10}$$

Then by testing (3.10) with $\partial_t \Delta u$ we have

$$\begin{aligned} &\|\partial_t \Delta u\|_2^2 + \frac{\nu}{2} \frac{d}{dt} (\|\Delta^2 u\|_2^2) \le \|\partial_t \Delta u\|_2 \|\Delta^2 (u^3 - u)\|_2 \\ \Longrightarrow\ &\|\partial_t \Delta u\|_2^2 + \frac{d}{dt} \|\Delta^2 u\|_2^2 \lesssim \|u\|_{H^4}^2 + \|u\|_{H^4}^6. \end{aligned}$$

As a result, the usual (integral) Grönwall's inequality shows that $\sup_{0\le t\le 1} \|u\|_{H^4} \lesssim 1$ since $s \ge 4$. We arrive at

$$\int_0^1 \|\partial_t \Delta u\|_2^2 \, \mathrm{dt} \lesssim \int_0^1 \|u\|_{H^4}^2 \, \mathrm{dt} \lesssim 1. \tag{3.11}$$

We then assume $T > 1$. Recall that from the energy conservation we can derive that

$$\frac{d}{dt}E(u) = \int_{\mathbb{T}^2} \partial_t u(-\nu\Delta u + f(u)) \, \mathrm{dx} = -\Big(\partial_t u, (-\Delta)^{-1}\partial_t u\Big) = -\||\nabla|^{-1}\partial_t u\|_2^2. \tag{3.12}$$

Then by integrating in time, it is clear that

$$\int_0^\infty \||\nabla|^{-1}\partial_t u\|_2^2 \, \mathrm{dt} \lesssim 1. \tag{3.13}$$

Therefore by Sobolev and Hölder's inequalities we see that

$$\begin{aligned}\int_1^T \|\partial_t \Delta u\|_2^2 \, dt &\lesssim \int_1^T \||\nabla|^{-1}\partial_t u\|_2^{\frac{1}{2}} \|\partial_t \langle\nabla\rangle^3 u\|_2^{\frac{3}{2}} \, \mathrm{dt} \\ &\lesssim \int_1^T \||\nabla|^{-1}\partial_t u\|_2^{\frac{1}{2}} \, \mathrm{dt} \\ &\lesssim \int_1^T \||\nabla|^{-1}\partial_t u\|_2^2 \, \mathrm{dt} \cdot T^{\frac{3}{4}} \lesssim 1 + T,\end{aligned} \tag{3.14}$$

where we need $\sup_{t\ge 1} \|\partial_t \langle\nabla\rangle^3 u\|_2 \lesssim 1$ from the smoothing effect Lemma 3.2. □

4. Energy stability of the scheme (1.11)

Recall that the Cahn-Hilliard equation (CH) takes the following form:

$$\begin{cases}\partial_t u = -\nu\Delta^2 u + \Delta f(u), \\ u(x,0) = u_0.\end{cases}$$

Here $f(u) = u^3 - u$, and the spatial domain Ω is taken to be the two dimensional 2π-periodic torus $\mathbb{T}^2$. The corresponding energy is defined by $E(u) = \int_\Omega (\frac{\nu}{2}|\nabla u|^2 + F(u)) \, dx$, where $F(u) = \frac{1}{4}(u^2-1)^2$, an anti-derivative of $f(u)$. Recall that we consider the stabilized integrator method (1.11):

$$\begin{cases}\dfrac{u^{n+1} - u^n}{\tau} + S\Delta^2(u^{n+1} - u^n) = \dfrac{e^{-\nu\Delta^2\tau} - 1}{\tau} u^n + \dfrac{1 - e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(u^n), \\ u^0 = \Pi_N u_0.\end{cases} \tag{4.1}$$

Here Π_N is truncation of Fourier modes of L^2 functions to $|k|_\infty \le N$. Note that here $E^0 = E(\Pi_N u_0)$ while $E_0 = E(u_0)$ and in general $E_0 \ne E^0$. In particular the following statement holds.

Lemma 4.1. *Suppose $E^0 = E(\Pi_N u_0)$ and $E_0 = E(u_0)$ as defined above, the following inequality holds:*

$$\sup_N E(\Pi_N u_0) \lesssim 1 + E_0, \text{where } u_0 \in H^1(\mathbb{T}^2)\,.$$

Proof. We rewrite $\Pi_N u_0$ as $\frac{1}{(2\pi)^2}\sum_{|k|\le N}\widehat{u_0}(k)e^{ik\cdot x}$, namely the Dirichlet partial sum of u_0.

$$\|\nabla(\Pi_N u_0)\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{(2\pi)^2}\sum_{|k|\le N}|k|^2|\widehat{u_0}(k)|^2 \le \frac{1}{(2\pi)^2}\sum_{|k|\in\mathbb{Z}^2}|k|^2|\widehat{u_0}(k)|^2 = \|\nabla(u_0)\|^2_{L^2(\mathbb{T}^2)}\,.$$

On the potential energy part, by the Sobolev inequality Lemma 2.1, $\|u_0\|_{L^4(\mathbb{T}^2)} \lesssim \|u_0\|_{H^1(\mathbb{T}^2)}$, this shows $u_0 \in L^4(\mathbb{T}^2)$ and hence the Dirichlet partial sum $\Pi_N u_0$ converges to u_0 in $L^4(\mathbb{T}^2)$. Then $\lim_{N\to\infty}\|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} = \|u_0\|_{L^4(\mathbb{T}^2)}$, which leads to $\sup_N\|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} < \infty$. By the Uniform Boundedness Principle, we derive $\sup_N\|\Pi_N\| < \infty$, i.e. $\sup_N\|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} \le c\|u_0\|_{L^4(\mathbb{T}^2)}$ for an absolute constant c. Combining the two estimates above we finish the proof. □

We will prove Theorem 1.6 by induction. To start with, we test the equation by $\frac{\nu\Delta\tau}{e^{-\nu\tau\Delta^2}-1}(u^{n+1} - u^n)$. Then the LHS of (4.1) is

$$\begin{aligned} \text{LHS} =& \frac{1}{\tau}\left(u^{n+1}-u^n, \frac{\nu\Delta\tau}{e^{-\nu\tau\Delta^2}-1}(u^{n+1}-u^n)\right) \\ &+ S\left(\Delta^2(u^{n+1}-u^n), \frac{\nu\Delta\tau}{e^{-\nu\tau\Delta^2}-1}(u^{n+1}-u^n)\right) \qquad (4.2)\\ =& \left(u^{n+1}-u^n, \frac{-\nu\Delta}{1-e^{-\nu\tau\Delta^2}}(u^{n+1}-u^n)\right) + S\left(u^{n+1}-u^n, \frac{-\nu\Delta^3\tau}{1-e^{-\nu\tau\Delta^2}}(u^{n+1}-u^n)\right), \qquad (4.3)\end{aligned}$$

where $(\cdot,\cdot)$ is the usual L^2 inner product. We observe that both the two operators $A = \frac{-\nu\Delta}{1-e^{-\nu\tau\Delta^2}}$ and $B = \frac{-\nu\Delta^3\tau}{1-e^{-\nu\tau\Delta^2}}$ are positive semi-definite and self-adjoint, therefore we have

$$\text{LHS} = \|\sqrt{A}(u^{n+1}-u^n)\|_2^2 + S\|\sqrt{B}(u^{n+1}-u^n)\|_2^2, \qquad (4.4)$$

where the operator $\sqrt{A}$ and $\sqrt{B}$ correspond to the following Fourier symbols:

$$\widehat{\sqrt{A}u}(k) = \sqrt{\frac{\nu|k|^2}{1-e^{-\nu|k|^4\tau}}}\,\widehat{u}(k); \quad \widehat{\sqrt{B}u}(k) = \sqrt{\frac{\nu|k|^6\tau}{1-e^{-\nu|k|^4\tau}}}\,\widehat{u}(k). \qquad (4.5)$$

To proceed from here, it is important to examine the behaviors of these two operators. Indeed we state the following lemma.

Lemma 4.2. *Suppose the operators $\sqrt{A}$ and $\sqrt{B}$ are defined in (4.5) above. Then the following inequalities hold for any function $u \in H^3(\mathbb{T}^2)$ and any $v \in H^1(\mathbb{T}^2)$ with zero-mean:*

$$\left(\frac{1}{\tau}\right)^{\frac{1}{2}} \||\nabla|^{-1}v\|_2 \le \|\sqrt{A}v\|_2, \quad \|\nabla u\|_2 \le \|\sqrt{B}u\|_2. \tag{4.6}$$

Proof. The (4.6) can be proved from the Fourier side. In particular, by the Parseval's identity we have

$$\|\sqrt{A}v\|_2^2 = \frac{1}{(2\pi)^2}\sum_{|k|\ge 1}\frac{\nu|k|^2}{1-e^{-\nu|k|^4\tau}}|\widehat{v}(k)|^2, \tag{4.7}$$

therefore it suffices to show

$$\frac{\nu|k|^2}{1-e^{-\nu|k|^4\tau}} \ge \frac{1}{\tau|k|^2}, \quad \text{or} \quad \frac{\nu\tau|k|^4}{1-e^{-\nu\tau|k|^4}} \ge 1. \tag{4.8}$$

To show (4.8), we can define an 1D function for $x \ge 1$, $h(x) = \frac{\nu\tau x^4}{1-e^{-\nu\tau x^4}} - 1$. By direct computation we can show $h'(x) > 0$ for $x \ge 1$ and $h(1) > 0$ noting $\nu\tau > 0$. Therefore (4.8) is proved. Then (4.7) implies

$$\frac{1}{\tau}\||\nabla|^{-1}v\|_2^2 \le \|\sqrt{A}v\|_2^2. \tag{4.9}$$

By the same reason we can show $\|\nabla u\|_2 \le \|\sqrt{B}u\|_2$. We hereby conclude (4.6). □

We then continue the proof of Theorem 1.6. We now focus on the RHS of (4.1):

$$\text{RHS} \tag{4.10}$$

$$=\left(\frac{e^{-\nu\Delta^2\tau}-1}{\tau}u^n, \frac{-\nu\Delta\tau}{1-e^{-\nu\tau\Delta^2}}(u^{n+1}-u^n)\right)$$

$$+\left(\frac{1-e^{-\nu\Delta^2\tau}}{\nu\tau\Delta}\Pi_N f(u^n), \frac{-\nu\Delta\tau}{1-e^{-\nu\tau\Delta^2}}(u^{n+1}-u^n)\right) \tag{4.11}$$

$$=\nu\left(u^n, \Delta(u^{n+1}-u^n)\right) - \left(\Pi_N f(u^n), u^{n+1}-u^n\right). \tag{4.12}$$

Note that $u^{n+1}, u^n \in X_N$, we have

$$\text{RHS} = \nu\left(\nabla u^n, \nabla(u^n - u^{n+1})\right) - \left(f(u^n), u^{n+1}-u^n\right). \tag{4.13}$$

Using the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$, we derive from (4.13) that

$$\text{RHS} = \frac{\nu}{2}(\|\nabla u^n\|_2^2 - \|\nabla u^{n+1}\|_2^2 + \|\nabla u^n - \nabla u^{n+1}\|_2^2) - (f(u^n), u^{n+1}-u^n). \tag{4.14}$$

To proceed further, by the fundamental theorem of calculus we arrive at

$$
\begin{aligned}
F(u^{n+1}) - F(u^n) &= f(u^n)(u^{n+1} - u^n) + \int_{u^n}^{u^{n+1}} f'(\sigma)(u^{n+1} - \sigma)\ \mathrm{d}\sigma \\
&= f(u^n)(u^{n+1} - u^n) + \int_{u^n}^{u^{n+1}} (3\sigma^2 - 1)(u^{n+1} - \sigma)\ \mathrm{d}\sigma \\
&= f(u^n)(u^{n+1} - u^n) + \frac{1}{4}(u^{n+1} - u^n)^2 \left(3(u^n)^2 + (u^{n+1})^2 + 2u^n u^{n+1} - 2\right).
\end{aligned}
\tag{4.15}
$$

Combining the estimates (4.4), (4.14) and (4.15) we arrive at

$$
\begin{aligned}
&\|\sqrt{A}(u^{n+1} - u^n)\|_2^2 + S\|\sqrt{B}(u^{n+1} - u^n)\|_2^2 + E(u^{n+1}) - E(u^n) \\
=&\frac{\nu}{2}\|\nabla(u^{n+1} - u^n)\|_2^2 + \frac{1}{4}\left((u^{n+1} - u^n)^2, 3(u^n)^2 + (u^{n+1})^2 + 2u^n u^{n+1} - 2\right).
\end{aligned}
\tag{4.16}
$$

By further considering (4.6) and (4.16) we indeed have

$$
\begin{aligned}
&\frac{1}{\tau}\||\nabla|^{-1}(u^{n+1} - u^n)\|_2^2 + (S - \frac{\nu}{2})\|\nabla(u^{n+1} - u^n)\|_2^2 + E(u^{n+1}) - E(u^n) \\
&\le \frac{1}{4}\left((u^{n+1} - u^n)^2, 3(u^n)^2 + (u^{n+1})^2 + 2u^n u^{n+1} - 2\right).
\end{aligned}
\tag{4.17}
$$

Moreover by Lemma 2.4, we derive that

$$
\begin{aligned}
&(\sqrt{\frac{2S}{\tau}} + \frac{S}{2} - \frac{\nu}{2})\|u^{n+1} - u^n\|_2^2 + E(u^{n+1}) - E(u^n) \\
\le&\frac{1}{4}\left((u^{n+1} - u^n)^2, 3(u^n)^2 + (u^{n+1})^2 + 2u^n u^{n+1} - 2\right).
\end{aligned}
\tag{4.18}
$$

Therefore to show the energy dissipation, it suffices to take

$$
\sqrt{\frac{2S}{\tau}} + \frac{S}{2} - \frac{\nu}{2} \ge \frac{3}{2}\max\left\{\|u^n\|_\infty^2\ ,\ \|u^{n+1}\|_\infty^2\right\}.
\tag{4.19}
$$

We rewrite the numerical scheme (1.11) as follows:

$$
u^{n+1} = \frac{S\tau\Delta^2 + e^{-\nu\Delta^2\tau}}{1 + S\tau\Delta^2}u^n + \frac{1 - e^{-\nu\Delta^2\tau}}{(1 + S\tau\Delta^2)\nu\Delta}\Pi_N f(u^n)
\tag{4.20}
$$

By the interpolation lemma (Lemma 2.5), to control $\|u^{n+1}\|_\infty$ and $\|u^n\|_\infty$, we can only focus on the $\dot{H}^1$-norm and $\dot{H}^{\frac{3}{2}}$-norm.

Lemma 4.3. *There is an absolute constant $c_1 > 0$ such that for any $n \geq 0$*

$$\begin{cases} \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} & \leq c_1 \cdot \left(\sqrt{\frac{N}{\nu}} + \frac{1}{S}\right) \cdot (E^n + 1) \\ \|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} & \leq \left(\frac{1}{S} + \frac{1}{S}\|u^n\|_\infty^2\right) \cdot \|u^n\|_{L^2(\mathbb{T}^2)} + \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} \, . \end{cases} \tag{4.21}$$

Proof of Lemma 4.3. It suffices to consider Fourier modes $1 \leq |k| \leq N$ from the Fourier side. We then obtain that

$$\begin{cases} \dfrac{(S\tau|k|^4 + e^{-\nu\tau|k|^4})|k|^{\frac{3}{2}}}{1 + S\tau|k|^4} \leq |k| \cdot N^{\frac{1}{2}} \\ \dfrac{(1 - e^{-\nu\tau|k|^4})|k|^{\frac{3}{2}}}{(1 + S\tau|k|^4)\nu|k|^2} \leq \dfrac{1}{S|k|^{\frac{1}{2}}}, \end{cases} \tag{4.22}$$

where we have observed that $1 - e^{-\nu\tau|k|^4} \leq \nu\tau|k|^4$. It then follows by (4.20) that

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \leq N^{\frac{1}{2}}\|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{S}\||\nabla|^{-\frac{1}{2}} f(u^n)\|_{L^2(\mathbb{T}^2)}. \tag{4.23}$$

Here the notation $|\nabla|^s = (-\Delta)^{\frac{s}{2}}$, corresponds to the Fourier side $|k|^s$. Note that by the Sobolev inequality Lemma 2.1 we have

$$\begin{aligned} \|\langle\nabla\rangle^{-\frac{1}{2}} f(u^n)\|_{L^2(\mathbb{T}^2)} &\lesssim \|f(u^n)\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} = \|(u^n)^3 - u^n\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} \\ &= \left(\int_{\mathbb{T}^2} ((u^n)^3 - u^n)^{\frac{4}{3}} \,\mathrm{dx}\right)^{\frac{3}{4}} \\ &\lesssim E^n + 1 \, . \end{aligned}$$

Therefore (4.23) can be estimated by the following:

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\sqrt{\frac{N}{\nu}} + \frac{1}{S}\right)(E^n + 1).$$

Similarly, we get

$$\begin{cases} \dfrac{(S\tau|k|^4 + e^{-\nu\tau|k|^4})|k|}{1 + S\tau|k|^4} \leq |k| \\ \dfrac{(1 - e^{-\nu\tau|k|^4})|k|}{(1 + S\tau|k|^4)\nu|k|^2} \leq \dfrac{\nu\tau|k|^4}{S\tau\nu|k|^6}|k| = \dfrac{1}{S|k|} \, . \end{cases} \tag{4.24}$$

It then follows by (4.24) that

$$
\begin{aligned}
\|u^{n+1}\|_{\dot H^1(\mathbb{T}^2)} &\le \|u^n\|_{\dot H^1(\mathbb{T}^2)} + \frac{1}{S}\|f(u^n)\|_{L^2(\mathbb{T}^2)} \\
&\le \|u^n\|_{\dot H^1(\mathbb{T}^2)} + \frac{1}{S}\|(1-(u^n)^2)\cdot u^n\|_{L^2(\mathbb{T}^2)} \\
&\le \|u^n\|_{\dot H^1(\mathbb{T}^2)} + (\frac{1}{S} + \frac{\|u^n\|_\infty^2}{S})\|u^n\|_{L^2(\mathbb{T}^2)}.
\end{aligned}
$$

We then complete the proof of Lemma 4.3. □

Proof of Theorem 1.6. Now we will complete the proof for Theorem 1.6 by induction:

Step 1: The induction $n \to n+1$ step. Assuming $E^n \le E^{n-1} \le \cdots \le E^0$ and $E^n \le \sup_N E(\Pi_N u_0)$, we will show $E^{n+1} \le E^n$. This implies $\|u^n\|_{\dot H^1}^2 = \|\nabla u^n\|_{L^2}^2 \le \frac{2E^n}{\nu} \le \frac{2E^0}{\nu}$. By Lemma 2.5 and adopting the notation $f \lesssim_{E^0} g$ to denote that $f \le C(E^0)\cdot g$ for some constant $C(E^0)$ depending only on E^0, we have after applying (2.6)

$$
\begin{aligned}
\|u^n\|_\infty^2 &\lesssim \|u^n\|_{\dot H^1}^2 \left(\sqrt{\log(3 + c_1\left(\sqrt{\frac{N}{\nu}} + \frac{1}{S}\right)(E^{n-1}+1))}\right)^2 + 1 \\
&\lesssim \frac{2E^0}{\nu}\left(1 + \log(S) + \log(\frac{1}{\nu}) + (\log(N))\right) + 1 \\
&\lesssim_{E^0} \nu^{-1}\left(1 + \log(S) + \log(\frac{1}{\nu}) + \log(N)\right) + 1\,.
\end{aligned}
\tag{4.25}
$$

Define $m_0 := \nu^{-1}(1 + \log(S) + |\log(\nu)| + \log(N))$, and note that $E^0 \le \sup_N E(\Pi_N u_0) \lesssim E_0 + 1$, the inequality above (4.25) is then rewritten as follows:

$$
\|u^n\|_\infty^2 \lesssim_{E_0} m_0 + 1.
$$

On the other hand by Lemma 4.3 and Lemma 2.5,

$$
\begin{aligned}
\|u^{n+1}\|_\infty^2 &\lesssim 1 + \|u^{n+1}\|_{\dot H^1}^2 \log(3 + \|u^{n+1}\|_{\dot H^{\frac{3}{2}}}) \\
&\lesssim 1 + (1 + \frac{1 + \|u^n\|_\infty^2}{S})^2 \|u^n\|_{\dot H^1}^2 \log(3 + \|u^{n+1}\|_{\dot H^{\frac{3}{2}}}) \\
&\lesssim_{E_0} 1 + (1 + \frac{m_0 + 1}{S})^2 \frac{1}{\nu} \log(3 + \|u^{n+1}\|_{\dot H^{\frac{3}{2}}}) \\
&\lesssim_{E_0} 1 + (1 + \frac{m_0}{S})^2 m_0 \\
&\lesssim_{E_0} 1 + \frac{m_0^3}{S^2} + m_0.
\end{aligned}
\tag{4.26}
$$

The sufficient condition (4.19) thus becomes

$$\begin{cases} \sqrt{\dfrac{2S}{\tau}} + \dfrac{S}{2} - \dfrac{\nu}{2} \geq C(E_0)\left(1 + m_0 + \dfrac{m_0^3}{S^2}\right), \\ m_0 = \nu^{-1}\left(1 + \log(S) + |\log(\nu)| + \log(N)\right) . \end{cases}$$

It then suffices for us to choose S such that

$$S \gg_{E_0} m_0 + \nu = \nu^{-1}\left(1 + \log(S) + |\log(\nu)| + \log(N)\right) + \nu,$$

where the notation $B \gg_{E_0} D$ means there exists a large constant C_{E_0} depending only on E_0 such that $B \geq C_{E_0} D$. In fact, for $\nu \gtrsim 1$, we can take $S \gg_{E_0} \nu + \log(N)$; if $0 < \nu \ll 1$, we will choose $S = C_{E_0} \cdot \nu^{-1}(|\log \nu| + \log N)$, where C_{E_0} is a large constant depending only on E_0. Therefore it suffices to choose

$$S = C_{E_0} \nu^{-1}(\log N + |\log \nu|) + \nu. \tag{4.27}$$

Step 2: We now check the induction base step $n = 0$. Clearly we only need to check

$$\sqrt{\frac{2S}{\tau}} + \frac{S}{2} - \frac{\nu}{2} \geq \frac{3}{2}\max\{\|\Pi_N u_0\|_\infty^2, \|u^1\|_\infty^2\}.$$

By Lemma 4.3,

$$\begin{aligned} \|u^1\|_{\dot{H}^1} &\leq \left(1 + \frac{1}{S} + \frac{1}{S}\|\Pi_N u_0\|_\infty^2\right) \cdot \|u_0\|_{\dot{H}^1} \\ &\leq \left(1 + \frac{1}{S} + \frac{1}{S}\|\Pi_N u_0\|_\infty^2\right) \cdot \sqrt{\frac{2E^0}{\nu}} . \end{aligned}$$

As a result,

$$\begin{aligned} \|u^1\|_\infty &\lesssim 1 + \|u^1\|_{\dot{H}^1}\sqrt{\log(3 + \|u^1\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim 1 + \left(1 + \frac{1}{S} + \frac{1}{S}\|\Pi_N u_0\|_\infty^2\right)\sqrt{\frac{2E^0}{\nu}}\sqrt{\log\left(3 + c_1\left(\sqrt{\frac{S}{\nu}} + \frac{1}{S}\right)(E_0 + 1)\right)} \\ &\lesssim_{E_0} 1 + \left(1 + \frac{1}{S} + \frac{1}{S}\|\Pi_N u_0\|_\infty^2\right) \cdot \nu^{-\frac{1}{2}} \cdot \sqrt{1 + \log(S) + |\log(\nu)| + \log N}. \end{aligned}$$

Thus we need to choose S such that

$$\begin{aligned} \sqrt{\frac{2S}{\tau}} + \frac{S}{2} - \frac{\nu}{2} \geq C_{E_0} \cdot \left(1 + \frac{1}{S} + \frac{1}{S}\|\Pi_N u_0\|_\infty^2\right)^2 \cdot \nu^{-1} \\ \cdot \left(1 + \log(S) + |\log(\nu)| + \log N\right), \end{aligned} \tag{4.28}$$

and

$$\sqrt{\frac{2S}{\tau}} + \frac{S}{2} - \frac{\nu}{2} \geq \frac{3}{2}\|\Pi_N u_0\|_\infty^2, \tag{4.29}$$

where C_{E_0} is a large constant depending only on E_0. Note that by Morrey's inequality,

$$\|\Pi_N u_0\|_{L^\infty(\mathbb{T}^2)} \lesssim \|\Pi_N u_0\|_{H^2(\mathbb{T}^2)} \lesssim \|u_0\|_{H^2(\mathbb{T}^2)}.$$

Then it suffices to take S such that

$$S \gg_{E_0} \|u_0\|_{H^2}^2 + \nu^{-1}(|\log(\nu)| + \log N) + \nu\ . \tag{4.30}$$

This completes the induction and hence proves the theorem. □

5. L^2 error estimate for the scheme (1.11)

In this section, we will study the L^2 error between the numerical solution and the exact PDE solution and eventually prove Theorem 1.8. To start with, we consider the auxiliary L^2 error estimate for near solutions. As will be seen later, we will iterate the auxiliary error estimate carefully in order to lower the regularity assumption.

5.1. Auxiliary L^2 error estimate for near solutions

Consider the following auxiliary system u^n and v^n in X_N (we refer the readers to Section 2 for the definition of X_N) for the first order scheme:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} + S\Delta^2(u^{n+1}-u^n) = \dfrac{e^{-\nu\Delta^2\tau}-1}{\tau}u^n + \dfrac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(u^n) + \Delta G_n^1 \\ \dfrac{v^{n+1}-v^n}{\tau} + S\Delta^2(v^{n+1}-v^n) = \dfrac{e^{-\nu\Delta^2\tau}-1}{\tau}v^n + \dfrac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(v^n) + \Delta G_n^2 \\ u^0 = u_0,\ v^0 = v_0, \end{cases} \tag{5.1}$$

where G_n^1 and G_n^2 are two fixed functions in X_N. We define that $G_n = G_n^1 - G_n^2$.

Proposition 5.1. *For solutions of (5.1) with initial data u_0 and v_0 of zero mean, assume for some $N_1 > 0$,*

$$\sup_{n\geq 0}\|\nabla u^n\|_2 + \sup_{n\geq 0}\|v^n\|_\infty + \sup_{n\geq 0}\|\nabla v^n\|_2 \leq N_1\ . \tag{5.2}$$

Then for any $m \geq 1$,

$$\begin{aligned}\|u^m - v^m\|_2^2 \leq &\exp\left(m\tau\cdot\frac{C(1+N_1^4)}{\nu}\right)\cdot\Big(\|Lu_0 - Lv_0\|_2^2 + S\tau\|Pu_0 - Pv_0\|_2^2 \\ &+ \frac{C}{\nu}\tau\sum_{n=0}^{m-1}\|LG_n\|_2^2 + C\nu\tau^2\sum_{n=0}^{m-1}\|\Delta^2(u^{n+1}-v^{n+1})\|_2^2\Big),\end{aligned} \tag{5.3}$$

where $C>0$ is an absolute constant and L, P are defined from the Fourier side as below:

$$\begin{cases} \widehat{Lu}(k) = \sqrt{\frac{\nu|k|^4\tau}{1-e^{-\nu|k|^4\tau}}}\widehat{u}(k) \\ \widehat{Pu}(k) = \sqrt{\frac{\nu|k|^8\tau}{1-e^{-\nu|k|^4\tau}}}\widehat{u}(k). \end{cases} \tag{5.4}$$

Lemma 5.2. *We remark here by similar arguments in Lemma 4.2 we have for $g, h \in C^\infty(\mathbb{T}^2)$ that the following holds:*

$$\|\Delta g\|_2 \le \|Pg\|_2 \lesssim \|\Delta g\|_2 + \sqrt{\tau}\|\Delta^2 g\|_2, \quad \|h\|_2 \le \|Lh\|_2 \lesssim \|h\|_2 + \sqrt{\tau}\|\Delta h\|_2. \tag{5.5}$$

Proof of Proposition 5.1. We assume Lemma 5.2 for the time being. Write $\varepsilon^n = u^n - v^n$. Then

$$\frac{\varepsilon^{n+1}-\varepsilon^n}{\tau} + S\Delta^2(\varepsilon^{n+1}-\varepsilon^n) = \frac{e^{-\nu\Delta^2\tau}-1}{\tau}\varepsilon^n + \frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N(f(u^n)-f(v^n)) + \Delta G_n\,. \tag{5.6}$$

Taking L^2-inner product with $\frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}$ on both sides of (5.6) and recalling similar computations in previous section, we have the LHS of (5.6) is

$$\begin{aligned} \text{LHS} =& \left(\frac{\varepsilon^{n+1}-\varepsilon^n}{\tau}, \frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}\right) + \left(S\Delta^2(\varepsilon^{n+1}-\varepsilon^n), \frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}\right) \\ =& \frac{1}{2\tau}(\|L\varepsilon^{n+1}\|_2^2 - \|L\varepsilon^n\|_2^2 + \|L(\varepsilon^{n+1}-\varepsilon^n)\|_2^2) \\ &+ \frac{S}{2}(\|P\varepsilon^{n+1}\|_2^2 - \|P\varepsilon^n\|_2^2 + \|P(\varepsilon^{n+1}-\varepsilon^n)\|_2^2). \end{aligned} \tag{5.7}$$

Similarly, the RHS of (5.6) can be given as below

$$\begin{aligned} \text{RHS} =& \left(\frac{e^{-\nu\Delta^2\tau}-1}{\tau}\varepsilon^n + \Delta G_n + \frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N(f(u^n)-f(v^n)), \frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}\right) \\ :=& I_1 + I_2 + I_3, \end{aligned} \tag{5.8}$$

where $(.\ ,\ .)$ denotes the L^2 inner product. We then estimate different parts I_1, I_2 and I_3 as follows.

$$\begin{aligned} I_1 =& \left(\frac{e^{-\nu\Delta^2\tau}-1}{\tau}\varepsilon^n, \frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}\right) \\ =& -\nu(\Delta\varepsilon^n, \Delta\varepsilon^{n+1}) \\ =& -\nu(\Delta\varepsilon^{n+1}, \Delta\varepsilon^{n+1}) + \nu(\Delta\varepsilon^{n+1}, \Delta\varepsilon^{n+1}-\Delta\varepsilon^n) \\ =& -\frac{\nu}{2}(\|\Delta\varepsilon^{n+1}\|_2^2 + \|\Delta\varepsilon^n\|_2^2) + \frac{\nu}{2}\|\Delta\varepsilon^{n+1}-\Delta\varepsilon^n\|_2^2. \end{aligned} \tag{5.9}$$

Similarly, by Hölder's inequality we obtain that

$$I_2 = \left(\Delta G_n, \frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}\right) \le \|P\varepsilon^{n+1}\|_2\|LG_n\|_2 \le \frac{C}{\nu}\|LG_n\|_2^2 + \frac{\nu}{100}\|P\varepsilon^{n+1}\|_2^2. \quad (5.10)$$

On the other hand, note that Π_N is a self-adjoint operator $(\Pi_N f,\ g) = (f,\ \Pi_N g)$, since it is just an N-th Fourier mode truncation. Therefore we have

$$I_3 = \left(\frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N(f(u^n)-f(v^n)), \frac{\nu\Delta^2\tau}{1-e^{-\nu\Delta^2\tau}}\varepsilon^{n+1}\right) = \left(f(u^n)-f(v^n), \Delta\Pi_N\varepsilon^{n+1}\right). \quad (5.11)$$

Then by the fundamental theorem of calculus, we have

$$\begin{aligned} f(u^n)-f(v^n) &= \int_0^1 f'(v^n+s\varepsilon^n)ds\cdot\varepsilon^n \\ &= (a_1+a_2(v^n)^2)\varepsilon^n + a_3v^n(\varepsilon^n)^2 + a_4(\varepsilon^n)^3\ , \end{aligned} \quad (5.12)$$

where a_i are exact constants which can be computed. Note that we will denote C to be an absolute constant whose value may vary in different lines. To proceed further we indeed have by (5.2) that

$$\begin{aligned} |\left((a_1+a_2(v^n)^2)\varepsilon^n\ ,\ \Delta\Pi_N\varepsilon^{n+1}\right)| &\le C(1+\|v^n\|_\infty^2)\|\Delta\varepsilon^{n+1}\|_2\|\varepsilon^n\|_2 \\ &\le \frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2. \end{aligned} \quad (5.13)$$

Moreover, the other two terms can be estimated similarly. In particular, recalling (2.2) that for f defined in $\mathbb{T}^2$ with zero mean one has $\|f\|_4^2 \lesssim \|f\|_2\|\nabla f\|_2$. Therefore recalling that $\varepsilon^n = u^n - v^n$ is of zero mean, we have

$$\begin{aligned} |\left(a_3v^n(\varepsilon^n)^2\ , \Delta\Pi_N\varepsilon^{n+1}\right)| &\le C\|v^n\|_\infty\|\varepsilon^n\|_4^2\|\Delta\varepsilon^{n+1}\|_2 \le \frac{CN_1^2}{\nu}\|\varepsilon^n\|_4^4 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2 \\ &\le C\frac{N_1^2}{\nu}\|\varepsilon^n\|_2^2\|\nabla\varepsilon^n\|_2^2 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2 \le \frac{CN_1^4}{\nu}\|\varepsilon^n\|_2^2 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2. \end{aligned} \quad (5.14)$$

Similarly, recalling (2.3) that we also have

$$\begin{aligned} |\left(a_4(\varepsilon^n)^3\ , \Delta\Pi_N\varepsilon^{n+1}\right)| &\le C\|\varepsilon^n\|_6^3\|\Delta\varepsilon^{n+1}\|_2 \le \frac{C}{\nu}\|\varepsilon^n\|_6^6 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2 \\ &\le \frac{C}{\nu}\|\varepsilon^n\|_2^2\|\nabla\varepsilon^n\|_2^4 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2 \\ &\le \frac{CN_1^4}{\nu}\|\varepsilon^n\|_2^2 + \frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2. \end{aligned} \quad (5.15)$$

One can then conclude from (5.12), (5.13), (5.14) and (5.15) that

$$|I_3| = |\left(f(u^n)-f(v^n), \Delta\Pi_N\varepsilon^{n+1}\right)| \le \frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2 + \frac{3\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2. \quad (5.16)$$

Collecting the estimates (5.8), (5.9), (5.10) and (5.16) (and noting that $-\frac{\nu}{2}\|\Delta\varepsilon^n\|_2^2$ is negative), we get

$$\begin{aligned}\text{RHS} \le & -\frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2 + \frac{\nu}{2}\|\Delta\varepsilon^{n+1} - \Delta\varepsilon^n\|_2^2 + \frac{C}{\nu}\|LG_n\|_2^2 \\ & + \frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2 + \frac{\nu}{100}\|P\varepsilon^{n+1}\|_2^2.\end{aligned} \tag{5.17}$$

Taking the same choice of S as in Theorem 1.6, it then follows by (5.7) and (5.17) that

$$\begin{aligned}&\frac{\|L\varepsilon^{n+1}\|_2^2 - \|L\varepsilon^n\|_2^2}{\tau} + S(\|P\varepsilon^{n+1}\|_2^2 - \|P\varepsilon^n\|_2^2) \\ \le & \frac{C}{\nu}\|LG_n\|_2^2 + \frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2 + C\nu\tau\|\Delta^2\varepsilon^{n+1}\|_2^2,\end{aligned} \tag{5.18}$$

where C is an absolute positive constant that can be computed exactly and the last inequality is due to Lemma 5.2: $-\frac{\nu}{8}\|\Delta\varepsilon^{n+1}\|_2^2 + \frac{\nu}{100}\|P\varepsilon^{n+1}\|_2^2 \le C\nu\tau\|\Delta^2\varepsilon^{n+1}\|_2^2$. Then by defining

$$\begin{cases} y_n = \|L\varepsilon^n\|_2^2 + \tau S\|P\varepsilon^n\|_2^2, \\ \alpha = \frac{C(1+N_1^4)}{\nu}, \\ \beta_n = \frac{C}{\nu}\|LG_n\|_2^2 + C\nu\tau\|\Delta^2\varepsilon^{n+1}\|_2^2, \end{cases} \tag{5.19}$$

(5.18) thus can be formulated into

$$\frac{y_{n+1} - y_n}{\tau} \le \alpha y_n + \beta_n .$$

Applying discrete Grönwall's inequality Lemma 2.6, we arrive at

$$\begin{aligned}&\|u^m - v^m\|_2^2 = \|\varepsilon^m\|_2^2 \le y_m \\ \le & \exp\left(m\tau\cdot\frac{C(1+N_1^4)}{\nu}\right)\cdot\Big(\|Lu_0 - Lv_0\|_2^2 + S\tau\|Pu_0 - Pv_0\|_2^2 \\ & + \frac{C}{\nu}\tau\sum_{n=0}^{m-1}\|LG_n\|_2^2 + C\nu\tau^2\sum_{n=0}^{m-1}\|\Delta^2\varepsilon^{n+1}\|_2^2\Big).\end{aligned} \tag{5.20}$$

This completes the proof. □

Remark 5.3. The proof of Proposition 5.1 only requires the uniform boundedness of $\|\nabla u^n\|_2 + \|\nabla v^n\|_2$ rather than $\|\Delta u^n\|_2 + \|\Delta v^n\|_2$. It somehow makes sense by considering the equation (1.11) where the nonlinear part requires lower regularity assumption and the linear part can be controlled by its sign.

Proof of Lemma 5.2. First of all the same arguments from Lemma 4.2 give

$$\|\Delta g\|_2 \le \|Pg\|_2, \quad \|h\|_2 \le \|Lh\|_2.$$

To see the rest, we consider the Fourier series of Lh again:

$$\begin{aligned}\|Lh\|_2^2 =& \frac{1}{(2\pi)^2}\sum_{|k|\ge 1}\frac{\nu|k|^4\tau}{1-e^{-\nu|k|^4\tau}}|\widehat{h}(k)|^2 \\ =& \frac{1}{(2\pi)^2}\Big(\sum_{\nu|k|^4\tau\le\log 2}+\sum_{\nu|k|^4\tau\ge\log 2}\frac{\nu|k|^4\tau}{1-e^{-\nu|k|^4\tau}}|\widehat{h}(k)|^2\Big) \\ \lesssim& \sum_{\nu|k|^4\tau\le\log 2}\frac{\nu|k|^4\tau}{1-e^{-\nu|k|^4\tau}}|\widehat{h}(k)|^2+\tau\sum_{\nu|k|^4\tau\ge\log 2}|k|^4|\widehat{h}(k)|^2.\end{aligned} \tag{5.21}$$

The high frequency part can be controlled by $\|\Delta h\|_2$ by using the standard Parseval's identity. For the lower frequency part, we then consider the function $\phi(x)=\frac{x}{1-e^{-x}}$. By direct computation, we see that $\phi'(x)>0$ for $x>0$ which shows that $\phi(x)$ is monotone increasing. Therefore we have

$$\sum_{\nu|k|^4\tau\le\log 2}\frac{\nu|k|^4\tau}{1-e^{-\nu|k|^4\tau}}|\widehat{h}(k)|^2 \lesssim \sum_{|k|\ge 1}|\widehat{h}(k)|^2. \tag{5.22}$$

Collecting the estimates (5.21) and (5.22) we then get the desired result. The same reason gives

$$\|Pg\|_2 \lesssim \|\Delta g\|_2+\sqrt{\tau}\|\Delta^2 g\|_2. \quad \square$$

Remark 5.4. If we in addition assume $\max_{1\le n\le M}\|\Delta^2 u^n\|_2+\|\Delta^2 v^n\|_2\le N_2$, where M is a fixed iteration number, then Proposition 5.1 implies

$$\begin{aligned}&\|u^m-v^m\|_2^2=\|\varepsilon^m\|_2^2 \\ &\le \exp\left(m\tau\cdot\frac{C(1+N_1^4)}{\nu}\right)\cdot\Big(\|Lu_0-Lv_0\|_2^2+S\tau\|Pu_0-Pv_0\|_2^2 \\ &+\frac{C}{\nu}\tau\sum_{n=0}^{m-1}\|LG_n\|_2^2+Cm\nu\tau^2N_2^2\Big).\end{aligned} \tag{5.23}$$

Indeed if $v^n=u(t_n)$ is the continuous solution later in the error analysis, then $\max_{1\le n\le M}\|\Delta^2 v^n\|_2\le\sup_{t\ge 0}\|u(t)\|_{H^4(\mathbb{T}^2)}\lesssim 1$ by Lemma 3.1. On the other hand to see the uniform boundedness of the discrete solution $\max_{1\le n\le M}\|\Delta^2 u^n\|_2\lesssim 1$, we introduce the following Lemma.

Lemma 5.5. *Assume $\{u^n\}$ is the discrete solution to our LGI method (1.10) or (1.11):*

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} + S\Delta^2(u^{n+1}-u^n) = \dfrac{e^{-\nu\Delta^2\tau}-1}{\tau}u^n + \dfrac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N[f(u^n)], \\ u^0 = \Pi_N u_0, \end{cases}$$

where the initial data $u_0 \in H^k(\mathbb{T}^2)$ with $k \geq 4$. Then we have

$$\max_{1\leq n\leq M} \|\Delta^2 u^n\|_2 \lesssim 1,$$

where M is the iteration number.

Proof of Lemma 5.5. Recalling from (1.11) then we have

$$\Delta^2(u^{n+1}-u^n) + S\tau\Delta^4(u^{n+1}-u^n) = \Delta^2(e^{-\nu\Delta^2\tau}-1)u^n + \Delta\frac{1-e^{-\nu\Delta^2\tau}}{\nu}\Pi_N f(u^n). \tag{5.24}$$

Then we multiply (5.24) with $\Delta^2(u^{n+1}-u^n)$ and integrate to obtain that

$$\|\Delta^2(u^{n+1}-u^n)\|_2 \lesssim \|\Delta^2 u^n\|_2 + \|\Delta f(u^n)\|_2. \tag{5.25}$$

Then by a bootstrap argument we have $\|\Delta^2 u^{n+1}\|_2 \leq \|\Delta^2(u^{n+1}-u^n)\|_2 + \|\Delta^2 u^n\|_2 \lesssim 1$ for any $n = 0, 1, 2, \cdots, M$ where M is the fixed iteration number. This completes the proof. □

Remark 5.6. As a consequence of Remark 5.4 and Lemma 5.5, if we further have

$$\max_{1\leq n\leq M} \|u^n\|_{H^s} + \|v^n\|_{H^s} \lesssim 1 \tag{5.26}$$

with $s \geq 4$ then we have a better estimate than in Proposition 5.1 as in Proposition 5.7 below through an iterative Grönwall framework. Indeed (5.26) holds as long as the initial data $u_0 \in H^s$ for the same $s \geq 4$. The proof of (5.26) is similar to the proof of Lemma 5.5. Note that here the constants in (5.26) and Lemma 5.5 depend on M but M is a fixed iteration number that can be understood as T/τ if the time interval $(0, T)$ is given.

Proposition 5.7. *For solutions of (5.1), for any m satisfying $1 \leq m \leq M$ and any $s \geq 4$ we have the following inequality:*

$$\begin{aligned} &\|u^m - v^m\|_2^2 = \|\varepsilon^m\|_2^2 \\ &\leq \exp\left(m\tau \cdot \frac{C(1+\nu^2+N_1^4)}{\nu}\right) \cdot \Big(\|Lu_0 - Lv_0\|_2^2 + S\tau\|Pu_0 - Pv_0\|_2^2 \\ &\quad + \frac{C}{\nu}\tau\sum_{n=0}^{m-1}\|LG_n\|_2^2 + Cm\nu\tau^{1+\frac{s}{4}}\Big). \end{aligned} \tag{5.27}$$

Proof of Proposition 5.7. Similar to (5.18) we also have

$$\frac{\|L\varepsilon^{n+1}\|_2^2-\|L\varepsilon^n\|_2^2}{\tau}+S(\|P\varepsilon^{n+1}\|_2^2-\|P\varepsilon^n\|_2^2)$$
$$\leq\frac{C}{\nu}\|LG_n\|_2^2+\frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2+C\nu\tau\|\Delta^2\varepsilon^n\|_2^2,$$

which is true with help of extra $S\|P\varepsilon^{n+1}-P\varepsilon^n\|_2^2$ from the LHS (5.7). Note that from the Fourier side we have for $s\geq 4$ that

$$\begin{aligned}\|\Delta^2\varepsilon^n\|_2^2\lesssim&\sum_{|k|\leq J}|k|^8|\widehat{\varepsilon^n}(k)|^2+\sum_{|k|\geq J}|k|^8|\widehat{\varepsilon^n}(k)|^2\\ \lesssim&J^8\sum_{|k|\leq J}|\widehat{\varepsilon^n}(k)|^2+J^{-2(s-4)}\sum_{|k|\geq J}|k|^{2s}|\widehat{\varepsilon^n}(k)|^2\\ \lesssim&J^8\|\varepsilon^n\|_2^2+J^{-2(s-4)}\|\varepsilon^n\|_{H^s}^2,\end{aligned}$$

where J is to be determined. As a result, we can derive that

$$\begin{aligned}&\frac{\|L\varepsilon^{n+1}\|_2^2-\|L\varepsilon^n\|_2^2}{\tau}+S(\|P\varepsilon^{n+1}\|_2^2-\|P\varepsilon^n\|_2^2)\\ \leq&\frac{C}{\nu}\|LG_n\|_2^2+\frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2+C\nu\tau(J^8\|\varepsilon^n\|_2^2+J^{-2(s-4)}),\end{aligned}\tag{5.28}$$

where $\max_{1\leq n\leq M}\|\varepsilon^n\|_{H^s}\lesssim 1$ because of Lemma 3.1 and Remark 5.6 (Lemma 5.5). Now by choosing $J^8\approx\frac{1}{\tau}$ we can further get

$$\begin{aligned}&\frac{\|L\varepsilon^{n+1}\|_2^2-\|L\varepsilon^n\|_2^2}{\tau}+S(\|P\varepsilon^{n+1}\|_2^2-\|P\varepsilon^n\|_2^2)\\ \leq&\frac{C}{\nu}\|LG_n\|_2^2+\frac{C(1+N_1^4)}{\nu}\|\varepsilon^n\|_2^2+C\nu\|\varepsilon^n\|_2^2+C\nu\tau^{\frac{s}{4}}.\end{aligned}\tag{5.29}$$

Then by defining

$$\begin{cases}y_n=\|L\varepsilon^n\|_2^2+\tau S\|P\varepsilon^n\|_2^2,\\ \alpha=\frac{C(1+N_1^4)}{\nu}+C\nu,\\ \beta_n=\frac{C}{\nu}\|LG_n\|_2^2+C\nu\tau^{\frac{s}{4}},\end{cases}\tag{5.30}$$

we obtain (5.27) and finish the proof of Proposition 5.7. □

5.2. *L^2 error estimate of the 2D Cahn-Hilliard equation*

In this section, to simplify the notation, we will write $x\lesssim y$ if $x\leq C(\nu,u_0)\ y$ for a constant C depending on ν and u_0. We consider the system

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau}+S\Delta^2(u^{n+1}-u^n)=\dfrac{e^{-\nu\Delta^2\tau}-1}{\tau}u^n+\dfrac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(u^n) \\ \partial_t u=-\nu\Delta^2 u+\Delta(f(u)) \\ u^0=\Pi_N u_0,\ u(0)=u_0, \end{cases} \tag{5.31}$$

where u^n is the discrete numerical solution at n-th time step while $u(x,t)$ is the exact solution to (CH). In order to prove Theorem 1.8, it is clear that we shall estimate G_n introduced in (5.1) from the previous proposition. Now recall that the continuous equation can be rewritten in the mild form (Duhamel' formula):

$$\begin{aligned} u(t_{n+1}) =& e^{-\nu\tau\Delta^2}u(t_n)+\int_{t_n}^{t_{n+1}} e^{-\nu(t_{n+1}-\sigma)\Delta^2}\Delta f(u(\sigma))\ d\sigma \\ =& e^{-\nu\tau\Delta^2}u(t_n)+\frac{1-e^{-\nu\tau\Delta^2}}{\nu\Delta}f(u(t_n))+\int_{t_n}^{t_{n+1}} e^{-\nu(t_{n+1}-\sigma)\Delta^2}\Delta(f(u(\sigma))-f(u(t_n)))\ d\sigma\,. \end{aligned} \tag{5.32}$$

This implies that

$$\begin{aligned} &\frac{u(t_{n+1})-u(t_n)}{\tau} \\ =&\frac{e^{-\nu\tau\Delta^2}-1}{\tau}u(t_n)+\frac{1-e^{-\nu\tau\Delta^2}}{\nu\tau\Delta}f(u(t_n))+\frac{1}{\tau}\int_{t_n}^{t_{n+1}} e^{-\nu(t_{n+1}-\sigma)\Delta^2}\Delta(f(u(\sigma))-f(u(t_n)))\ d\sigma \\ =&\frac{e^{-\nu\tau\Delta^2}-1}{\tau}u(t_n)+\frac{1-e^{-\nu\tau\Delta^2}}{\nu\tau\Delta}f(u(t_n))+\Delta\frac{1}{\tau}\int_{t_n}^{t_{n+1}} e^{-\nu(t_{n+1}-\sigma)\Delta^2}(f(u(\sigma))-f(u(t_n)))\ d\sigma\,. \end{aligned} \tag{5.33}$$

We then rewrite (5.33) into the following form:

$$\begin{aligned} \frac{u(t_{n+1})-u(t_n)}{\tau}+S\Delta^2(u(t_{n+1})-u(t_n)) =& \frac{e^{-\nu\Delta^2\tau}-1}{\tau}u(t_n)+\frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(u(t_n)) \\ &+\Delta\Pi_{>N}\frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta^2}f(u)(t_n)+\Delta Q_n\,, \end{aligned} \tag{5.34}$$

where $\Pi_{>N}=id-\Pi_N$, the large mode truncation operator and

$$Q_n=\frac{1}{\tau}\int_{t_n}^{t_{n+1}} e^{-\nu(t_{n+1}-\sigma)\Delta^2}(f(u(\sigma))-f(u(t_n)))\ d\sigma+S\int_{t_n}^{t_{n+1}}\partial_t\Delta u\ d\sigma\,. \tag{5.35}$$

For the sake of simplicity, we denote H to be the following quantity:

$$H_n := \Pi_{>N} \frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta^2} f(u)(t_n). \tag{5.36}$$

In order to prove Theorem 1.8, we now estimate $\|LQ_n\|_2$ and $\|LH_n\|_2$ in order to apply Proposition 5.1, Proposition 5.7 and Remark 5.6.

Proof of Theorem 1.8. By Theorem 1.6 and Lemma 3.1, $\sup_{n\geq 0}\|u^n\|_{H^1} \lesssim 1$ and $\sup_{t>0}\|u(t)\|_{H^s} \lesssim 1$ for $s \geq 6$ given in the statement. Thus the assumptions of Proposition 5.1 and Proposition 5.7 (auxiliary L^2 error estimate proposition) are satisfied for (5.31) and (5.34). Recall that (5.35)

$$Q_n = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} e^{-\nu(t_{n+1}-\sigma)\Delta^2}(f(u(\sigma)) - f(u(t_n)))\, d\sigma + S\int_{t_n}^{t_{n+1}} \partial_t\Delta u \, d\sigma .$$

Recalling (5.5) then we can estimate as

$$\|LQ_n\|_2 \lesssim \|Q_n\|_2 + \sqrt{\tau}\|\Delta Q_n\|_2.$$

We first estimate $\|Q_n\|_2$.

$$\begin{aligned}\|Q_n\|_2 &\lesssim \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \|f(u(\sigma)) - f(u(t_n)\|_2 \, d\sigma + S\int_{t_n}^{t_{n+1}} \|\partial_t\Delta u\|_2 \, d\sigma \\ &\lesssim S\int_{t_n}^{t_{n+1}} \|\partial_t\Delta u\|_2 \, d\sigma + \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, d\sigma \cdot \|f'(u)\|_{L_t^\infty L_x^\infty}.\end{aligned} \tag{5.37}$$

Note that by Lemma 3.1,

$$\|\partial_t u\|_2 + \|f'(u)\|_\infty \leq \nu\|\Delta^2 u\|_2 + \|\Delta(u^3-u)\|_2 + \|3u^2-1\|_\infty \lesssim 1. \tag{5.38}$$

Here we require that $\partial_t u \in L^2$ or $u \in H^4$. On the other hand, due to the smoothing effect and energy dissipation (we refer the readers to Proposition 3.3), we have

$$\int_0^{t_m} \|\partial_t\Delta u\|_2^2 \lesssim 1 + t_m,$$

as long as $s \geq 4$. Therefore we can estimate (5.37) as follows (for $t_m \geq 1$):

$$\sum_{n=0}^{m-1} \|Q_n\|_2^2 \lesssim (1+S)^2\tau \cdot (1+t_m) . \tag{5.39}$$

For the rest terms $\sqrt{\tau}\|\Delta Q_n\|_2$ we can obtain very similar results as long as $s \geq 6$ and thereby we can conclude that:

$$\sum_{n=0}^{m-1} \|LQ_n\|_2^2 \lesssim (1+S)^2(\tau+\tau^2)\cdot(1+t_m). \tag{5.40}$$

We now focus on the $\|LH_n\|_2$ term. Firstly recall that $\|LH_n\|_2^2 \lesssim \|H_n\|_2^2 + \tau\|\Delta H_n\|_2^2$, we then estimate $\|H_n\|_2^2$ below. From the Fourier side we can infer that

$$\begin{aligned}\|H_n\|_2^2 \leq \|\Pi_{>N} f(u)\|_2^2(t_n) &= \sum_{|k|>N} \left|\widehat{f(u)}(k)\right|^2 (t_n) \\ &\leq \sum_{|k|>N} |k|^{2s} \left|\widehat{f(u)}(k)\right|^2 (t_n)\cdot |k|^{-2s} \\ &\lesssim N^{-2s}\cdot \|f(u)\|_{H^s}^2(t_n) \\ &\lesssim N^{-2s}\,.\end{aligned} \tag{5.41}$$

Similarly, we get $\tau\|\Delta H_n\|_2^2 \lesssim \tau N^{-2s}$. As a result, assuming $\tau \lesssim 1$ or the time step is bounded (which is very natural) we get

$$\sum_{n=0}^{m-1} \|LH_n\|_2^2 \lesssim_s m\cdot(1+\tau^2)N^{-2s} \lesssim \frac{t_m N^{-2s}}{\tau}\,. \tag{5.42}$$

Therefore, combining (5.40) and (5.42) we can conclude that

$$\tau\sum_{n=0}^{m-1}\left(\|LQ_n\|_2^2 + \|LH_n\|_2^2\right) \lesssim_s (1+t_m)(\tau^2+N^{-2s})(1+S)^2\,. \tag{5.43}$$

In order to apply Proposition 5.1, it remains to control $\|Lu^0 - Lu(0))\|_2$ and $\|Pu^0 - Pu(0)\|_2$. Note that by (5.5) in Lemma 5.2 and similar arguments in (5.41) we have

$$\begin{aligned}\|Lu^0 - Lu(0)\|_2^2 = \|L\Pi_N u_0 - Lu_0\|_2^2 &\lesssim \|\Pi_N u_0 - u_0\|_2^2 + \tau\|\Delta(\Pi_N u_0 - u_0)\|_2^2 \\ &\lesssim N^{-2s} + \tau N^{-2(s-2)}.\end{aligned} \tag{5.44}$$

Similarly, we have

$$\|Pu^0 - Pu(0)\|_2^2 = \|P\Pi_N u_0 - Pu_0\|_2^2 \lesssim N^{-2(s-2)} + \tau N^{-2(s-4)}\,. \tag{5.45}$$

We now apply the auxiliary solutions estimate in Proposition 5.7 and Remark 5.6. Noting the estimates (5.43), (5.44) and (5.45) we can get

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+S)^2 e^{Ct_m}\left(N^{-2s} + \tau^2\cdot N^{-2(s-4)} + \tau^2 + \tau^{\frac{s}{4}}\right)\,.$$

Thus since we have

$$\|u^m - u(t_m)\|_2 \le (1+S)\cdot C_2 \cdot e^{C_1 t_m}\left(N^{-s} + \tau + \tau^{\frac{s}{8}} + \tau \cdot N^{-s+4}\right),$$

where $C_1 > 0$ is a constant depending on ν and u_0; $C_2 > 0$ is a constant depending on s, ν and u_0. This completes the proof of L^2 error estimate. □

Remark 5.8. We would like to emphasize here that the usual discrete Grönwall's inequality can yield the following result: assume for some $N_1 > 0$, for solutions of (5.1) we have

$$\sup_{n\ge 0}\|\nabla u^n\|_2 + \sup_{n\ge 0}\|v^n\|_\infty + \sup_{n\ge 0}\|\nabla v^n\|_2 \le N_1 . \tag{5.46}$$

Then for any $m \ge 1$,

$$\begin{aligned}&\|u^m - v^m\|_2^2 = \|\varepsilon^m\|_2^2 \le y_m \\ &\le \exp\left(m\tau \cdot \frac{C(1+N_1^4)}{\nu}\right)\cdot\left(\|Lu_0 - Lv_0\|_2^2 + S\tau\|Mu_0 - Mv_0\|_2^2 + \frac{4}{\nu}\tau\sum_{n=0}^{m-1}\|L^2 G_n\|_2^2\right),\end{aligned} \tag{5.47}$$

where $C > 0$ is an absolute constant. L, M are defined from the Fourier side as below:

$$\begin{cases} \widehat{Lu}(k) = \sqrt{\frac{\nu|k|^4\tau}{1-e^{-\nu|k|^4\tau}}}\,\widehat{u}(k) \\ \widehat{Mu}(k) = \sqrt{\frac{\nu|k|^8\tau}{1-e^{-\nu|k|^4\tau}}}\,\widehat{u}(k), \end{cases} \tag{5.48}$$

where L is the same operator defined in Proposition 5.1. The operator $L^2 = L\circ L$, i.e. L composite with L.

Nevertheless, this result together with the usual L^2 error analysis gives the usual convergence result for solutions of high regularity in an early version [19]. We list the result here without proof. We clearly see that Proposition 5.9 can be covered by Theorem 1.8 via our newly developed iterative discrete Grönwall framework.

Proposition 5.9 *(L^2 error estimate: the high regularity case). Fix $\nu > 0$ and let $u_0 \in H^s$, $s \ge 10$. Let $0 < \tau \le M$ for some $M > 0$. Let $u(t)$ be the continuous solution to the 2D Cahn-Hilliard equation with initial data u_0. Let u^m, $m \ge 1$ be defined in (1.11) with initial data u^0. Define $t_0 = 0$ and $t_m = m\tau$ for $m \ge 1$. Then for any $m \ge 1$,*

$$\|u^m - u(t_m)\|_2 \le (1+S)\cdot C_2 \cdot e^{C_1 t_m}\left(N^{-s} + \tau + \tau \cdot N^{-s+4}\right),$$

where $C_1 > 0$ is a constant depending on ν, u_0; $C_2 > 0$ is a constant depending on s, ν and u_0.

6. Discussion

In this manuscript we give a systematic approach on applying LGI-type methods to models where maximum principle no longer exists by solving the Cahn-Hilliard equation with a first order LGI method and showing the energy dissipation.

In particular, we prove the energy dissipation of LGI methods for the Cahn-Hilliard equation without assuming any strong Lipschitz condition or L^∞ boundedness. Furthermore, we also analyze the L^2 error and present some numerical simulations to demonstrate the dynamics. Indeed, the analysis framework can be applied to more general models and higher order schemes.

CRediT authorship contribution statement

X.C. wrote the main manuscript; X.C., Z.L. and S.W. did the math analysis; X.C. and S.W. did the programming. All authors reviewed the manuscript.

Funding

X.C. is partially supported by NSFC (Grant No. 12401270 and No. 42450192), Natural Science Foundation of Shanghai (Grant No. 24ZR1404200) and Shanghai Magnolia Talent Plan Pujiang Project (Grant No. 24PJA007). S.W. is partially supported by the China National Postdoctoral Program for Innovative Talents (Grant No. 20250061) and the China Postdoctoral Science Foundation (Grant No. 2024M760058).

Appendix A. Numerical evidence

In this section, we present several numerical results. To begin with we present numerical evidence that show the necessity of the stabilizers.

A.1. A benchmark computation with different S-values

In the first several experiments we vary the choice of the stabilizers, namely put $S = 0, 0.01, 0.1$. Moreover for the rest of the parameters, we choose $\nu = 0.01$, $\tau = 0.1$, $N_x = N_y = 256$ and the initial data u_0 is given basically "supported" in seven circles as below:

$$u_0(x, y) = -1 + \sum_{i=1}^{7} f_0\left(\sqrt{(x-x_i)^2 + (y-y_i)^2} - r_i\right), \tag{A.1}$$

where

$$f_0(s) = \begin{cases} 2e^{-\frac{\nu}{s^2}}, & \text{if } s < 0; \\ 0, & \text{otherwise.} \end{cases}$$

The centers and radii of the chosen circles are given in the Table 1 below.

The cases $S = 0$ (left) and $S = 0.01$ (right) can be found in Fig. 2 and the case $S = 0.1$ can be found in the left of Fig. 3. As you can tell, the choice of $S = 0.1$ can guarantee the energy dissipation already. Recall the discussion in Theorem 1.6 and Remark 1.7, we see that the optimal size of S could be much smaller than $\nu^{-1}|\log \nu|$ that we will not dig further in this direction.

Table 1
Choice of centers and radii.

x_i	$-\frac{\pi}{2}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$
y_i	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{3\pi}{4}$	0	$\frac{\pi}{2}$
r_i	$\frac{\pi}{5}$	$\frac{2\pi}{15}$	$\frac{2\pi}{15}$	$\frac{\pi}{10}$	$\frac{\pi}{10}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$

Table 2
Errors at $T = 0.5$ when τ varies. Here $\nu = 1$, $S = 0.1$ and $N_x = N_y = 256$.

$\tau = 0.01$	L^2-error	L^∞-error
τ	9.099e-05	1.156e-03
$\tau/2$	4.523e-05	5.745e-04
$\tau/4$	2.255e-05	2.864e-04
$\tau/8$	1.126e-05	1.430e-04
$\tau/16$	5.624e-06	7.143e-05
$\tau/32$	2.811e-06	3.570e-05

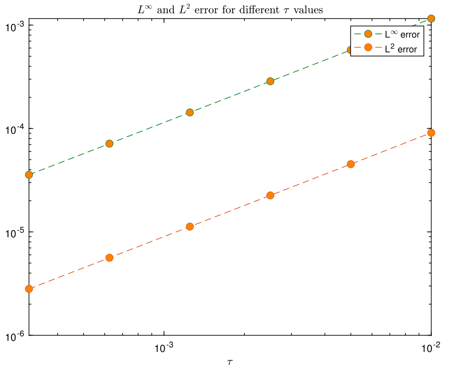


Fig. 1. Errors at $T = 0.5$ when τ varies. Here $\nu = 1$, $S = 0.1$ and $N_x = N_y = 256$.

A.2. Convergence test

In this subsection we consider a benchmark computation test with initial data being $u_0 = 0.5 * \sin(x)\sin(y)$. We take $\nu = 1$, $S = 0.1$ and $N_x = N_y = 256$. Then we consider the exact solution $u_e = 0.5 * e^{-t}\sin(x)\sin(y)$ corresponding to certain forcing term that can be computed explicitly. With these settings we perform our numerical experiments with various time steps $\tau = \frac{0.01}{2^k}$ with $k = 0, 1, ..., 6$. The relative L^2-errors and L^∞-errors at time $T = 0.5$ are presented below in Table 2. (See Fig. 1.) As usual, the experimental order of convergence is computed by comparing the errors of two consecutive refinements. Indeed the rate of convergence indicates the order of the error is $O(\tau)$.

A.3. More dynamics and energy evolution

In this section, we present more dynamics of our method 1.11 with patterns indicating the energy dissipation. In Fig. 4 we present the dynamics of 2D Cahn-Hilliard equation using LGI method (1.11) where $\nu = 0.01$, S=0.1, $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0 = 0.5\sin(x)\sin(y)$. In Fig. 5 we present the dynamics choosing $\nu = 0.01$, $S = 0.1$, $\tau = 0.001$, $N_x = N_y = 256$ and the initial data $u_0 = 0.05\sin(x)\sin(y)$. In Fig. 6 we present the dynamics where $\nu = 0.01$, $S = 0.1$, $\tau = 0.01$, $N_x = N_y = 256$. The values of the initial data u_0 are random between -1 and 1.

Appendix B. Proof of Proposition 5.9

For the sake of completeness we sketch the proof of Proposition 5.9 here.

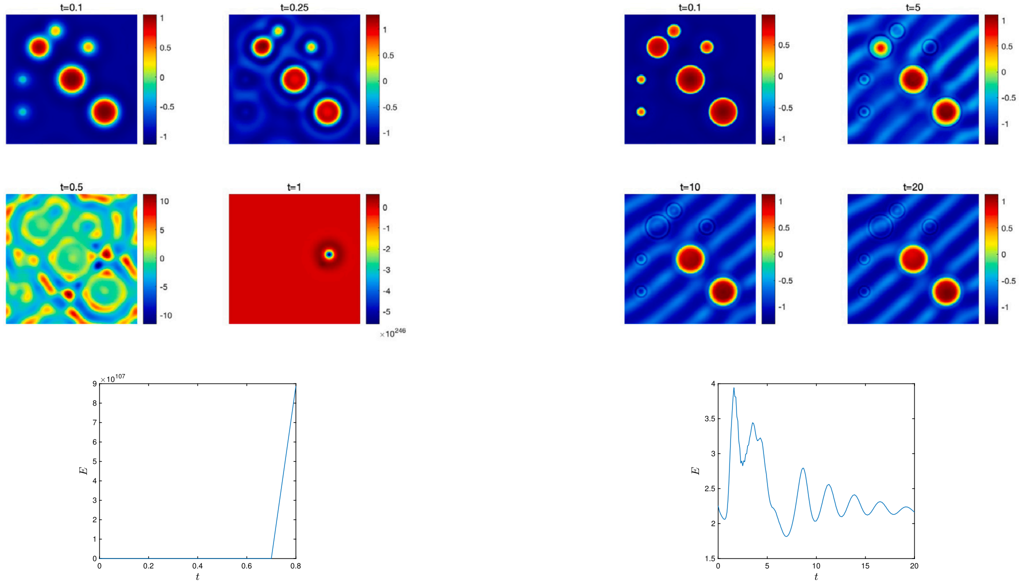


Fig. 2. Dynamics where $\nu = 0.01$, $\tau = 0.1$, $N_x = N_y = 256$ and the initial data u_0 is given in (A.1). $S = 0$ (Left), $S = 0.01$ (right). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

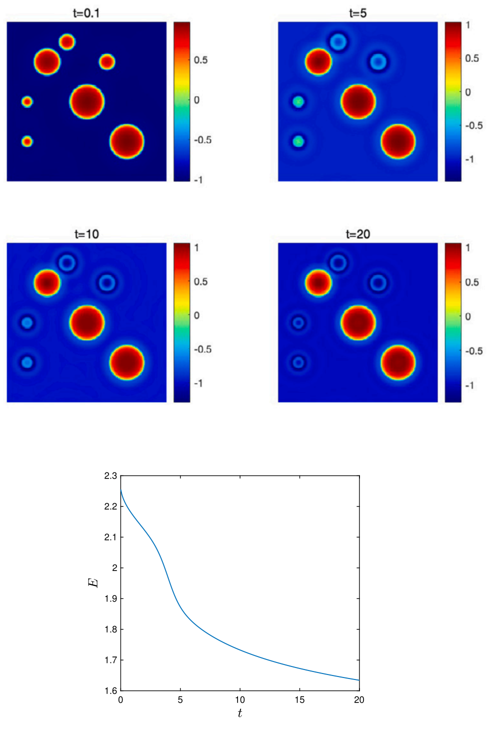


Fig. 3. Dynamics where $\nu = 0.01$, $S = 0.1$, $\tau = 0.1$, $N_x = N_y = 256$ and the initial data u_0 is given in (A.1).

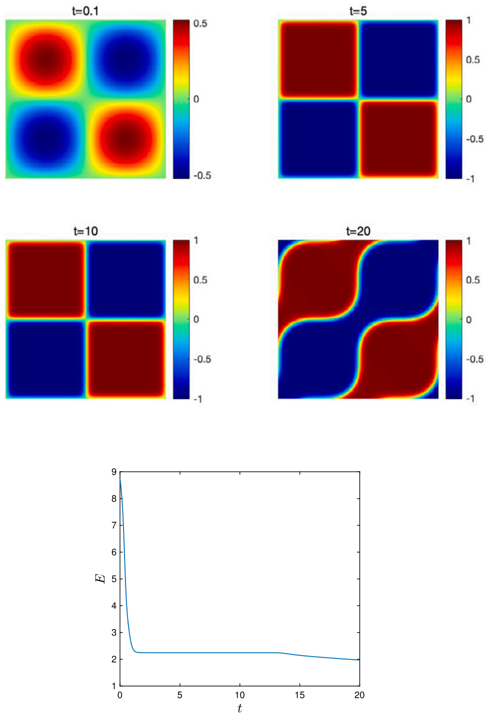


Fig. 4. Dynamics where $\nu = 0.01$, $S = 0.1$, $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0 = 0.5\sin(x)\sin(y)$.

Proof of Proposition 5.9. We first recall that for $g, h \in C^\infty(\mathbb{T}^2)$ that the following holds:

$$\|\Delta g\|_2 \le \|Mg\|_2 \lesssim \|\Delta g\|_2 + \sqrt{\tau}\|\Delta^2 g\|_2, \quad \|h\|_2 \le \|Lh\|_2 \lesssim \|h\|_2 + \sqrt{\tau}\|\Delta h\|_2. \tag{B.1}$$

We consider the system

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} + S\Delta^2(u^{n+1}-u^n) = \dfrac{e^{-\nu\Delta^2\tau}-1}{\tau}u^n + \dfrac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(u^n) \\ \partial_t u = -\nu\Delta^2 u + \Delta(f(u)) \\ u^0 = \Pi_N u_0\ ,\ u(0) = u_0\ . \end{cases} \tag{B.2}$$

Integrating the Cahn-Hilliard equation (CH) on the time interval $[t_n, t_{n+1}]$, we get

$$\begin{aligned} \frac{u(t_{n+1})-u(t_n)}{\tau} =& -\nu\Delta^2 u(t_n) - S\Delta^2\left(u(t_{n+1})-u(t_n)\right) \\ &+ \Delta\Pi_N f(u)(t_n) + \Delta\Pi_{>N} f(u)(t_n) + \Delta Q_n\ , \end{aligned} \tag{B.3}$$

where $\Pi_{>N} = id - \Pi_N$, the large mode truncation operator, and

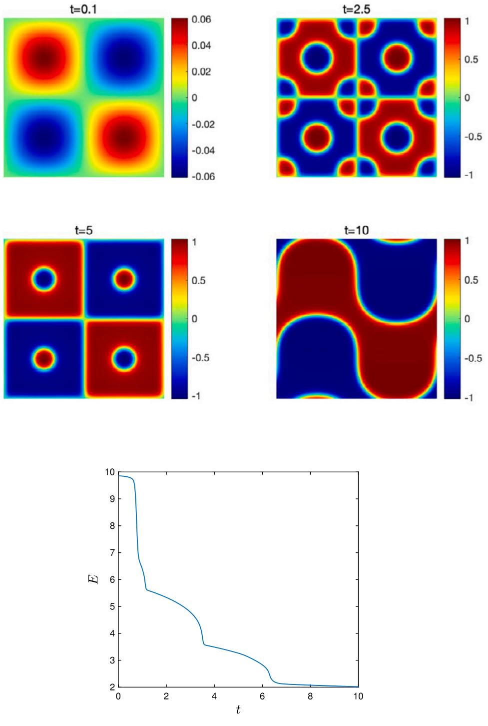


Fig. 5. Dynamics where $\nu = 0.01$, $S = 0.1$, $\tau = 0.001$, $N_x = N_y = 256$ and the initial data $u_0 = 0.05\sin(x)\sin(y)$.

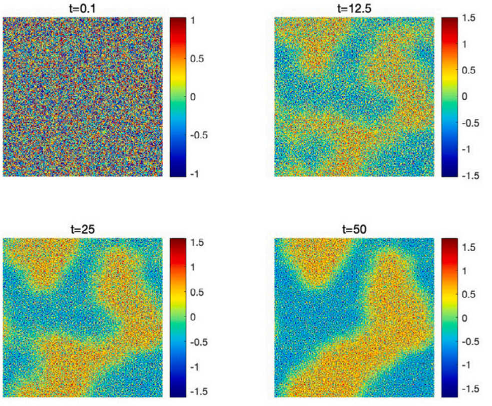
$$Q_n = -\frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t \Delta u \cdot (t_{n+1}-t)\ \mathrm{d}t + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t (f(u))(t_{n+1}-t)\ \mathrm{d}t + S\int_{t_n}^{t_{n+1}} \partial_t \Delta u\ \mathrm{d}t\ . \tag{B.4}$$

We then rewrite (B.3) into the following form:

$$\begin{aligned}
\frac{u(t_{n+1})-u(t_n)}{\tau} + S\Delta^2\left(u(t_{n+1})-u(t_n)\right) &= \frac{e^{-\nu\Delta^2\tau}-1}{\tau}u(t_n) + \frac{1-e^{-\nu\Delta^2\tau}}{\tau\nu\Delta}\Pi_N f(u(t_n)) \\
&+ \left(\frac{1-e^{-\nu\Delta^2\tau}}{\tau} - \nu\Delta^2\right)u(t_n) + \left(\Delta + \frac{e^{-\nu\Delta^2\tau}-1}{\nu\tau\Delta}\right)\Pi_N f(u(t_n)) \\
&+ \Delta\Pi_{>N} f(u)(t_n) + \Delta Q_n\ ,
\end{aligned} \tag{B.5}$$

For the sake of simplicity, we denote H_1, H_2, H_3 to be the following quantities:

$$\begin{cases}
H_1 := \left(\frac{1-e^{-\nu\Delta^2\tau}}{\tau\Delta} - \nu\Delta\right)u(t_n) = \frac{1-\nu\Delta^2\tau-e^{-\nu\Delta^2\tau}}{\tau\Delta}u(t_n), \\
H_2 := \left(1 + \frac{e^{-\nu\Delta^2\tau}-1}{\nu\tau\Delta^2}\right)\Pi_N f(u)(t_n), \\
H_3 := \Pi_{>N} f(u)(t_n).
\end{cases} \tag{B.6}$$



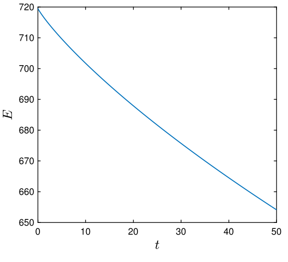


Fig. 6. Dynamics where $\nu = 0.01$, $S = 0.1$, $\tau = 0.01$, $N_x = N_y = 256$. The values of the initial data u_0 are random between -1 and 1.

By Remark 5.8 we now estimate $\|L^2 G_n\|_2$ by estimating $\|L^2 Q_n\|_2$ and $\|L^2 H_i\|_2$, $i = 1, 2, 3$.

By Theorem 1.6 and Lemma 3.1, $\sup_{n\geq 0} \|u^n\|_{H^1} \lesssim 1$ and $\sup_{t>0} \|u(t)\|_{H^s} \lesssim 1$ for $s \geq 10$ given in the statement. Thus the assumptions of Remark 5.8 are satisfied for (B.2) and (B.5). Recalling that we can control $\|L^2 Q_n\|_2 \lesssim \|Q_n\|_2 + \sqrt{\tau}\|\Delta Q_n\|_2 + \tau\|\Delta^2 Q_n\|_2$ as follows:

$$
\begin{aligned}
\|Q_n\|_2 &\lesssim \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, d\sigma + \int_{t_n}^{t_{n+1}} \|\partial_t (f(u))\|_2 \, d\sigma + S \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, d\sigma \\
&\lesssim (1+S) \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, d\sigma + \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, d\sigma \cdot \|f'(u)\|_{L_t^\infty L_x^\infty}.
\end{aligned}
\tag{B.7}
$$

Note that by Lemma 3.1,

$$
\|\partial_t u\|_2 + \|\partial_t \Delta u\|_2 + \|f'(u)\|_\infty \lesssim 1 .
$$

Therefore we can estimate (B.7) as follows

$$\int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, \mathrm{d}\sigma \lesssim \left(\int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \, \mathrm{d}\sigma \right)^{\frac{1}{2}} \cdot \sqrt{\tau} \, .$$

Therefore when $t_m \geq 1$, it is not hard to see that

$$\begin{aligned} \sum_{n=0}^{m-1} \|Q_n\|_2^2 &\lesssim \sum_{n=0}^{m-1} \left(\tau(1+S)^2 \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \, \mathrm{d}\sigma + \tau \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \, \mathrm{d}\sigma \right) \\ &\lesssim \tau(1+S)^2 \int_0^{t_m} \|\partial_t \Delta u\|_2^2 \, \mathrm{d}\sigma + \tau \int_0^{t_m} \|\partial_t u\|_2^2 \, \mathrm{d}\sigma \\ &\lesssim (1+S)^2 \tau \cdot (1+t_m) \, . \end{aligned} \tag{B.8}$$

For the rest terms $\sqrt{\tau}\|\Delta Q_n\|_2$ and $\tau\|\Delta^2 Q_n\|_2$ we will obtain very similar results as long as $s \geq 10$ (we need to estimate $\|\partial_t \Delta^3 u\|_2 \sim \|\Delta^5 u\|_2$ from $\|\Delta^2 Q_n\|_2$). Thereby we can conclude that:

$$\sum_{n=0}^{m-1} \|L^2 Q_n\|_2^2 \lesssim (1+S)^2 \tau \cdot (1+t_m). \tag{B.9}$$

We now focus on the $\|L^2 H_i\|_2$ terms. Firstly it is easy to see from the Fourier side that

$$\begin{aligned} \|L^2 H_1\|_2^2 =& \frac{1}{(2\pi)^2} \sum_{|k|\geq 1} \frac{\nu^2 |k|^4 (1-\nu|k|^4\tau - e^{-\nu|k|^4\tau})^2}{(1-e^{-\nu|k|^4\tau})^2} |\widehat{u}(k)|^2(t_n) \\ &\lesssim \sum_{\nu|k|^4\tau \leq c} + \sum_{\nu|k|^4\tau \geq c} \frac{|k|^4 (1-\nu|k|^4\tau - e^{-\nu|k|^4\tau})^2}{(1-e^{-\nu|k|^4\tau})^2} |\widehat{u}(k)|^2(t_n) \\ &:= J_1 + J_2. \end{aligned} \tag{B.10}$$

For the part where $\nu|k|^4\tau \geq c$, we indeed have

$$\begin{aligned} J_2 :=& \sum_{\nu|k|^4\tau \geq c} \frac{|k|^4 (1-\nu|k|^4\tau - e^{-\nu|k|^4\tau})^2}{(1-e^{-\nu|k|^4\tau})^2} |\widehat{u}(k)|^2(t_n) \\ &\lesssim \sum_{\nu|k|^4\tau \geq c} |k|^{12}\tau^2 |\widehat{u}(k)|^2(t_n) \\ &\lesssim \tau^2 \|\Delta^3 u\|_2^2(t_n). \end{aligned} \tag{B.11}$$

For the lower frequency part, we get

$$J_1 := \sum_{\nu|k|^4\tau\le c} \frac{|k|^4(1-\nu|k|^4\tau - e^{-\nu|k|^4\tau})^2}{(1-e^{-\nu|k|^4\tau})^2}|\widehat{u}(k)|^2(t_n)$$
$$\lesssim \sum_{1\le|k|\lesssim(\frac{1}{\tau})^{\frac14}} \frac{|k|^8(1-\nu|k|^4\tau - e^{-\nu|k|^4\tau})^2}{(1-e^{-\nu|k|^4\tau})^2}|\widehat{u}(k)|^2(t_n).$$

Recall that $\phi(x) = \frac{x}{1-e^{-x}}$ is monotone increasing for $x > 0$. Moreover, we have (by a simple observation) that

$$0 < \nu|k|^4\tau + e^{-\nu|k|^4\tau} - 1 < \nu^2|k|^8\tau^2. \tag{B.12}$$

Therefore we have

$$J_1 \lesssim \frac{1}{\tau^2} \sum_{1\le|k|\lesssim(\frac{1}{\tau})^{\frac14}} (\nu^2|k|^8\tau^2)^2|\widehat{u}(k)|^2(t_n) \lesssim \tau^2 \sum_{|k|\ge1} |k|^{16}|\widehat{u}(k)|^2(t_n) \lesssim \tau^2\|\Delta^4 u\|_2^2(t_n). \tag{B.13}$$

Combining (B.11)-(B.13), we get

$$\sum_{n=0}^{m-1} \|L^2 H_1\|_2^2 \lesssim \sum_{n=0}^{m-1} \tau^2(\|\Delta^3 u\|_2^2(t_n) + \|\Delta^4 u\|_2^2(t_n)) \lesssim m\tau^2 \lesssim (t_m+1)\cdot\tau. \tag{B.14}$$

Secondly we estimate $\|L^2 H_2\|_2^2$. Again by considering the Fourier expansion similar to (B.10), we arrive at

$$\begin{aligned}\|L^2 H_2\|_2^2 &= \frac{1}{(2\pi)^2} \sum_{1\le|k|\le N} \frac{(\nu\tau|k|^4 + e^{-\nu|k|^4\tau} - 1)^2}{(1-e^{-\nu\tau|k|^4})^2}|\widehat{f(u)}(k)|^2(t_n) \\ &\lesssim \sum_{\nu|k|^4\tau\le c} + \sum_{\nu|k|^4\tau\ge c} \frac{(\nu\tau|k|^4 + e^{-\nu|k|^4\tau} - 1)^2}{(1-e^{-\nu\tau|k|^4})^2}|\widehat{f(u)}(k)|^2(t_n).\end{aligned} \tag{B.15}$$

We can conclude from similar estimates that

$$\sum_{n=0}^{m-1} \|L^2 H_2\|_2^2 \lesssim m\tau^2 \lesssim (t_m+1)\cdot\tau. \tag{B.16}$$

Lastly we focus on $\|L^2 H_3\|_2^2$. Note that $\|L^2 H_3\|_2^2 \lesssim \|H_3\|_2^2 + \tau\|\Delta H_3\|_2^2 + \tau^2\|\Delta^2 H_3\|_2^2$, it follows that

$$\begin{aligned}\|H_3\|_2^2 = \|\Pi_{>N} f(u)\|_2^2(t_n) &= \sum_{|k|>N} \left|\widehat{f(u)}(k)\right|^2(t_n) \\ &\le \sum_{|k|>N} |k|^{2s}\left|\widehat{f(u)}(k)\right|^2(t_n)\cdot|k|^{-2s} \quad \lesssim N^{-2s}.\end{aligned} \tag{B.17}$$

Similarly, we get $\tau\|\Delta H_3\|_2^2 \lesssim \tau N^{-2s}$ and $\tau^2\|\Delta^2 H_3\|_2^2 \lesssim \tau^2 N^{-2s}$. As a result, assuming $\tau \lesssim 1$ or the time step is bounded (which is very natural) we get

$$\sum_{n=0}^{m-1} \|\Pi_{>N} f(u(t_n))\|_2^2 \lesssim_s m\cdot(1+\tau^2)N^{-2s} \lesssim \frac{t_m N^{-2s}}{\tau}\,. \tag{B.18}$$

Therefore, combining (B.9), (B.14), (B.16) and (B.18) we can conclude that

$$\tau\sum_{n=0}^{m-1}\left(\|L^2 Q_n\|_2^2 + \sum_{i=1,2,3}\|L^2 H_i\|_2^2\right) \lesssim_s (1+t_m)(\tau^2+N^{-2s})(1+S)^2\,. \tag{B.19}$$

It remains to control $\|Lu^0 - Lu(0))\|_2$ and $\|Mu^0 - Mu(0)\|_2$. Note that by (B.1) we have

$$\begin{aligned}\|Lu^0 - Lu(0)\|_2^2 &= \|L\Pi_N u_0 - Lu_0\|_2^2 \lesssim \|\Pi_N u_0 - u_0\|_2^2 + \tau\|\Delta(\Pi_N u_0 - u_0)\|_2^2\\ &\lesssim N^{-2s} + \tau N^{-2(s-2)}.\end{aligned} \tag{B.20}$$

Similarly, we have

$$\|Mu^0 - Mu(0)\|_2^2 = \|M\Pi_N u_0 - Mu_0\|_2^2 \lesssim N^{-2(s-2)} + \tau N^{-2(s-4)}\,. \tag{B.21}$$

Noting the estimates (B.19), (B.20) and (B.21) we can get

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+S)^2 e^{Ct_m}\left(N^{-2s} + \tau^2\cdot N^{-2(s-4)} + \tau^2\right)\,.$$

Thus

$$\|u^m - u(t_m)\|_2 \le (1+S)\cdot C_2\cdot e^{C_1 t_m}\left(N^{-s} + \tau + \tau\cdot N^{-s+4}\right)\,, \tag{B.22}$$

where $C_1 > 0$ is a constant depending on ν and u_0; $C_2 > 0$ is a constant depending on s , ν and u_0. □

Data availability

No data was used for the research described in the article.

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