THE VANISHING VISCOSITY LIMIT OF THE SURFACE QUASI-GEOSTROPHIC EQUATIONS IN BORDERLINE BESOV SPACES: RATES AND SAME TOPOLOGY CONVERGENCE

XINYU CHENG, ZHAONAN LUO, AND SHENG WANG

ABSTRACT. This paper resolves the inviscid limit problem for the dissipative surface quasi-geostrophic (SQG) equations in the borderline Besov space $B^1_{\infty,1}$. Due to the severely singular nature of the Riesz transform in the SQG velocity field, which is one full derivative more singular than the Navier-Stokes equations, standard energy methods suffer from catastrophic derivative loss. By introducing a delicate Bona-Smith type frequency truncation, we establish the local well-posedness uniformly in the viscosity parameter $\nu \in [0,1]$. Our main contribution is two-fold: we rigorously justify the vanishing viscosity limit with a sharp, quantitative convergence rate of $\mathcal{O}(\nu^{\min\{1,1/\gamma\}})$ in the lower-order topology $B^0_{\infty,1}$; more crucially, we prove that the strong convergence holds in the *same topology* ($B^1_{\infty,1}$) as the initial data. These results yield a complete and rigorous characterization of the inviscid dynamics for the highly singular SQG equations under borderline regularity.

MSCcodes: 35Q35, 35Q86,76B03, 35R11.

Keywords: Surface quasi-geostrophic equations; Inviscid limit; Besov spaces; Same topology convergence.

1. Introduction

1.1. Background and Model. In this paper, we are concerned with the dissipative (and inviscid) surface quasi-geostrophic (SQG) equations for the scalar field $\theta = \theta(x,t) : \mathbb{T}^2 \times [0,\infty) \to \mathbb{R}$:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = -\nu \Lambda^\gamma \theta, \\ u = \nabla^\perp \Lambda^{-1} \theta = (-\partial_2 \Lambda^{-1}\theta, \partial_1 \Lambda^{-1}\theta) := (-\mathcal{R}_2\theta, \mathcal{R}_1\theta); \\ \theta|_{t=0} = \theta_0, \end{cases} \tag{SQG}$$

where $\nu \geq 0$ represents the viscosity (or thermal diffusivity), $0 < \gamma \leq 2$ is the fractional dissipation parameter, and $\mathbb{T}^2 = [-\pi,\pi]^2$ is the standard two-dimensional periodic torus. The unknown θ denotes the potential temperature in geophysical fluid dynamics [14, 24]. The (SQG) system accurately models the evolution of temperature in a rapidly rotating stratified fluid and can be rigorously derived from the 3D Boussinesq equations [24].

Here, the operator $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ denotes the pair of Riesz transforms, and $\nabla^\perp = (-\partial_2, \partial_1)$. For $s \geq 0$, the fractional Laplacian $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ is defined via the Fourier transform: $\widehat{\Lambda^s\theta}(k) = |k|^s \widehat{\theta}(k)$ for $k \in \mathbb{Z}^2$. Without loss of generality, we consider solutions with zero mean, i.e., $\int_{\mathbb{T}^2} \theta(x,t)\,dx = 0$, which is conserved dynamically due to the incompressibility of the velocity field u.

The SQG equations exhibit profound mathematical analogies with the 2D incompressible Navier–Stokes (or Euler) equations. For the Navier–Stokes equations, the scalar vorticity ω satisfies a similar advection-diffusion equation, but with the velocity determined by the Biot-Savart law $u = -\nabla^\perp(-\Delta)^{-1}\omega$. In contrast, the SQG velocity is given by $u = \nabla^\perp \Lambda^{-1}\theta$, making the velocity one derivative more singular than the Navier–Stokes velocity. This severe non-local behavior makes the mathematical analysis of the SQG equations notoriously challenging, sharing features with the 3D Euler and Navier–Stokes equations.

1.2. Historical Review and Motivation. The well-posedness of fluid dynamics models has been a central topic in partial differential equations. The global regularity of the 3D Navier-Stokes and Euler equations remains one of the most formidable challenges, despite seminal classical results by Leray [19], Yudovich [26], and Beale-Kato-Majda [2]. The SQG equations have attracted equally extensive interest. For the SQG equations, Resnick [25] proved the global existence of weak solutions for $\nu \geq 0$ and $0 < \gamma \leq 2$ in $L^\infty_t L^2_x$. Marchand [22] obtained global weak solutions in

2 Sep 2026 19:53:02 PDT DiffEq+DynSys
260322-gks46 Version 3 - Submitted to Proc. Amer. Math. Soc.

$L_t^\infty H_x^{-1/2}$, and in $L_t^\infty L_x^p$ for $p \geq 4/3$ when $\nu > 0$. Further regularity and long-time asymptotic results for the SQG equations have been deeply investigated (see e.g., [8, 11, 21]). Ill-posedness in critical Sobolev spaces was recently established by Jeong-Kim [17], following the general framework of Bourgain and Li [3, 4]. Furthermore, the non-uniqueness of low-regularity solutions has been actively explored using convex integration schemes [5, 6, 10, 12, 16].

A particularly profound theoretical question is the behavior of solutions in the inviscid limit ($\nu \to 0$). Historically, establishing the inviscid limit and obtaining quantitative convergence rates were extensively studied for various fluid models (e.g., [9, 15, 23]), but typically restricted to lower-order topologies (e.g., L^2 or H^{s-1}). Establishing the inviscid limit in the *same topology* as the initial data represents a fundamental challenge due to severe derivative losses. For the Navier-Stokes to Euler transition, this was resolved in the highest-order Sobolev space H^s via delicate frequency truncation by Masmoudi [23], and subsequently extended to the critically embedded Besov space $B^1_{\infty,1}$ by Guo-Li-Yin [13].

Motivated by this progressive development, we aim to establish the same-topology inviscid limit for the SQG equations. However, this is highly non-trivial. For the Navier-Stokes equations, the velocity is determined by the Biot-Savart law $u = -\nabla^\perp(-\Delta)^{-1}\omega$. In contrast, the SQG velocity is given by $u = \nabla^\perp\Lambda^{-1}\theta$, making it one full derivative more singular. Consequently, analyzing the continuous dependence and the inviscid limit for SQG in the borderline Besov space $B^1_{\infty,1}$-the minimal space to guarantee $\nabla\theta \in L^\infty$ and uniquely define the Lagrangian flow-requires much more delicate handling of the highly singular Riesz transform $\mathcal{R}$.

1.3. **Main Results.** Our main theoretical contribution is a complete characterization of the local well-posedness uniformly in ν, alongside a sharp quantitative rate for the vanishing viscosity limit.

Theorem 1.1 (Local well-posedness and sharp inviscid limit)**.** Assume that $\nu \in [0,1]$ and $\gamma \in (0,2]$. For any $R > 0$ and initial data $\theta_0 \in B_R = \{\phi \in B^1_{\infty,1}(\mathbb{T}^2) : \|\phi\|_{B^1_{\infty,1}} \leq R,\ \int_{\mathbb{T}^2} \phi\, dx = 0\}$, there exists a time $T = T(R,\gamma) > 0$ such that the (SQG) equations admit a unique solution $\theta^\nu = S_T^\nu(\theta_0) \in C([0,T]; B^1_{\infty,1})$. Moreover, the following properties hold:

(i) Uniform bounds: There exists $C = C(R,\gamma) > 0$ such that
$$\|\theta^\nu\|_{L_T^\infty B^1_{\infty,1}} \leq C, \quad \forall\, \nu \in [0,1]. \tag{1.1}$$
Moreover, if $\theta_0 \in B^s_{\infty,1}$ for some $s > 1$, then $\exists\, C_1 = C_1(R,s,\gamma) > 0$
$$\|\theta^\nu(t)\|_{L_T^\infty B^s_{\infty,1}} \leq C_1\|\theta_0\|_{B^s_{\infty,1}}. \tag{1.2}$$

(ii) Continuous dependence: For every $\nu \in [0,1]$, the solution map
$$S_T^\nu : B_R \longrightarrow C([0,T]; B^1_{\infty,1})$$
is continuous, with a modulus uniform in ν. Namely, $\forall\, \eta > 0$, $\exists\, \delta = \delta(\eta,\theta_0,R,\gamma) > 0$ such that if $\phi_0 \in B_R$ and $\|\phi_0 - \theta_0\|_{B^1_{\infty,1}} < \delta$, then
$$\|\theta^\nu - \phi^\nu\|_{L_T^\infty B^1_{\infty,1}} < \eta, \quad \forall\, \nu \in [0,1]. \tag{1.3}$$

(iii) Sharp inviscid limit: Let θ^0 be the solution to the inviscid SQG equation ($\nu = 0$) with the same initial data θ_0. Then, the vanishing viscosity limit holds quantitatively in $B^0_{\infty,1}$:
$$\|\theta^\nu - \theta^0\|_{L_T^\infty B^0_{\infty,1}} \leq C\nu^{\min\{1,\frac{1}{\gamma}\}}. \tag{1.4}$$
Furthermore, in the high-norm topology, we have
$$\lim_{\nu\to 0}\|\theta^\nu - \theta^0\|_{L_T^\infty B^1_{\infty,1}} = 0. \tag{1.5}$$

Remark 1.2 (Convergence Rate vs. Same Topology Convergence)**.** It is conceptually crucial to distinguish between the convergence rate and the same-topology convergence in the context of hydrodynamic limits (cf. [7, 13, 23]). In the lower-order topology $B^0_{\infty,1}$, the error bound (1.4) provides a sharp, quantitative convergence rate dictated primarily by the physical viscous dissipation mechanism (i.e., $\mathcal{O}(\nu^{\min\{1,1/\gamma\}})$). Conversely, the strong convergence in the highest initial topology $B^1_{\infty,1}$ (1.5) represents the *same topology convergence*. At this critical level, the convergence speed is no longer solely governed by the viscosity, but is dominantly restricted by the high-frequency tail decay of the initial data (i.e., $\|(Id - S_N)\theta_0\|_{B^1_{\infty,1}}$).

Remark 1.3. The rate in (1.4) is sharp uniformly on bounded subsets of $B^1_{\infty,1}$. Indeed, fix $a>0$ and, for $k\in\mathbb{Z}^2\setminus\{0\}$, set $\theta_0^{(k)}(x)=a|k|^{-1}\cos(k\cdot x)$. The associated velocity is orthogonal to $\nabla\theta_0^{(k)}$, so the nonlinear term vanishes and

$$\theta^{\nu,(k)}(t)=e^{-\nu t|k|^\gamma}\theta_0^{(k)},\qquad \theta^{0,(k)}(t)=\theta_0^{(k)}.$$

Moreover, $\|\theta_0^{(k)}\|_{B^1_{\infty,1}}\simeq a$, so a may be chosen so that the entire family lies in any prescribed ball B_R. If $\gamma>1$, choosing $|k|\simeq\nu^{-1/\gamma}$ gives a $B^0_{\infty,1}$ error of order $\nu^{1/\gamma}$ at any fixed positive time. If $0<\gamma\le 1$, any fixed nonzero mode gives an error of order ν. Thus the exponent $\min\{1,1/\gamma\}$ cannot be improved for an estimate uniform on any B_R with $R>0$.

Remark 1.4. The exact rate of vanishing viscosity in $B^1_{\infty,1}$ of (1.5) is given below:

$$\|\theta^\nu-\theta^0\|_{L^\infty_T(B^1_{\infty,1})}\lesssim \nu 2^{\gamma N}+\nu^{\min\{1,\frac{1}{\gamma}\}}2^N+\|(Id-S_N)\theta_0\|_{B^1_{\infty,1}},$$

where S_N is a frequency cut-off operator. Note that by choosing N such that $2^{\gamma N}\sim\nu^{-\epsilon}$ for some $0<\epsilon<\min\{1,\gamma\}$one can derive (1.5).

Proposition 1.5. Let the assumptions in Theorem 1.1 hold. In addition, if the initial data $\theta_0\in B^\alpha_{\infty,1}$ with $1<\alpha\le 2$, then we have the following inviscid limit:

$$\|\theta^\nu-\theta^0\|_{L^\infty_T B^0_{\infty,1}}\le\begin{cases} C\nu, & \text{if}\quad \gamma\in(0,\alpha]\\ C\nu^{\frac{\alpha}{\gamma}}, & \text{if}\quad \gamma\in(\alpha,2],\end{cases}$$

for some constant $C>0$ depending on α, γ, T, and $\|\theta_0\|_{B^\alpha_{\infty,1}}$.

Remark 1.6 (Necessity and sharpness of the Besov framework)**.** The Sobolev estimates, that will be established in Proposition 1.7, clearly illustrate the limitations of the classical H^s framework. To effectively control the derivative loss inherent in the quasi-linear and non-local convection term, the Sobolev setting strictly requires the initial data to possess more than two spatial derivatives (i.e., $\theta_0\in H^s$ with $s>2$). This highlights the necessity and the sharpness of introducing the Besov framework. The space $B^1_{\infty,1}$ is a broader, critical space into which H^s embeds continuously for $s>2$. Remarkably, despite operating under significantly lower regularity conditions, Theorem 1.1 still gives the quantitative rate $\mathcal{O}(\nu^{\min\{1,1/\gamma\}})$ in the lower topology and convergence in the initial-data topology. This proves that the physical mechanism of the vanishing viscosity limit remains robust and mathematically stable, even when the initial data possesses only one derivative in the Besov sense.

Proposition 1.7 (Inviscid limit in H^s)**.** Let $s>2$ and $\gamma\in(0,2]$. For any $R>0$, assume that the initial data $\theta_0\in B_R=\{\phi\in H^s(\mathbb{T}^2):\|\phi\|_{H^s}\le R,\ \int_{\mathbb{T}^2}\phi\,dx=0\}$. Let θ^ν be the unique solution to the viscous (SQG) system ($\nu\in[0,1]$) and θ^0 be the solution to the inviscid SQG system ($\nu=0$). Then there exists $T>0$ such that the following estimates hold:

$$\|\theta^\nu-\theta^0\|_{L^\infty_T L^2}\le C\nu, \tag{1.6}$$

$$\|\theta^\nu-\theta^0\|_{L^\infty_T H^{s-1}}\le C\nu^{\min\{1,1/\gamma\}}, \tag{1.7}$$

$$\|\theta^\nu-\theta^0\|_{L^\infty_T H^{s-\gamma}}\le C\nu^{\min\{1,\gamma\}}, \tag{1.8}$$

$$\lim_{\nu\to 0}\|\theta^\nu-\theta^0\|_{L^\infty_T H^s}=0, \tag{1.9}$$

where $C>0$ depends on s, γ, $\|\theta_0\|_{H^s}$, and T.

The remainder of this paper is organized as follows. In Section 2, we introduce the necessary notations, the Littlewood-Paley decomposition, and collect crucial preliminary results, including the key commutator estimates and product rules in Besov and Sobolev spaces. Section 3 is devoted to the rigorous proof of our main results (Theorem 1.1), detailing the uniform bounds, the same-topology convergence, and the sharp quantitative vanishing viscosity limit within the borderline Besov framework $B^1_{\infty,1}$. In Section 4, we establish the continuous dependence and the inviscid limit in the classical Sobolev space H^s $(s>2)$ to provide a comparative perspective on the derivative loss mechanism. Finally, Section 5 offers concluding remarks, discussing the physical implications of our strict analytical framework for generalized rough data and vortex patch dynamics.

2. Notation and Preliminaries

2.1. **Notation.** Throughout this paper, for any two (non-negative in particular) quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context. We define $[A, B]$ to be $AB - BA$, namely the usual commutator.

Now we define the Littlewood-Paley dyadic decomposition on $\mathbb{T}^d$. Assume $\chi : [0, +\infty) \to [0, 1]$ is smooth, monotone, and satisfying

$$\chi(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}; \\ \frac{1}{2}, & x = \frac{3}{4}; \\ 0, & x \ge 1. \end{cases}$$

We then define $\varphi(x) = \chi(x) - \chi(2x)$, then

$$\chi(|x|) + \sum_{j \ge 1} \varphi\left(\frac{|x|}{2^j}\right) \equiv 1, \quad \forall x \in \mathbb{R}^d.$$

Definition 2.1. Assume f is a distribution on $\mathbb{T}^d$. Define the Littlewood-Paley dyadic decomposition of f by

$$\begin{cases} \Delta_0 f = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \chi(|k|) e^{ik\cdot x} = \hat{f}(0); \\ \Delta_j f = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \varphi\left(\frac{|k|}{2^j}\right) e^{ik\cdot x}, \ j \ge 1; \\ S_j f = \Delta_0 f + \sum_{1 \le l \le j} \Delta_l f, \ j \ge 1. \end{cases} \tag{2.1}$$

In particular, throughout this paper if we further assume f is of zero mean, then the Littlewood-Paley decomposition of f can be treated as below:

$$f = \sum_{j \ge 1} \Delta_j f. \tag{2.2}$$

Let $s \in \mathbb{R}, \ 1 \le p, r \le \infty$. The inhomogeneous Besov spaces $B^s_{p,r}$ is defined by

$$B^s_{p,r} = \{u \in S' : \|u\|_{B^s_{p,r}} = \left\|(2^{js}\|\Delta_j u\|_{L^p})_{j \ge 0}\right\|_{l^r} < \infty\}. \tag{2.3}$$

For zero-mean distributions on $\mathbb{T}^d$, the corresponding homogeneous and inhomogeneous Besov norms are equivalent; below we use the inhomogeneous notation.

2.2. **Preliminaries.** The Bernstein's inequality below is well known:

Lemma 2.2 (Bernstein's inequality)**.**

- **(i):** Let $p \in [1, \infty]$ and $s \in \mathbb{R}$. Then for any $j \in \mathbb{Z}$, we have
$$C\|\Lambda^s \Delta_j f\|_{L^p} \le 2^{js}\|\Delta_j f\|_{L^p} \le C'\|\Lambda^s \Delta_j f\|_{L^p}$$
with some constants C and C' depending only on p and s.
- **(ii):** Moreover, for $1 \le p \le q \le \infty$, we have
$$\|\Delta_j f\|_{L^q} \lesssim 2^{jd(\frac{1}{p} - \frac{1}{q})}\|\Delta_j f\|_{L^p},$$
where the constant depends only on p and q.

To keep the endpoint argument uniform over the full range $0 < \gamma \le 2$, we work directly in $B^1_{\infty,1}$ and do not pass to $p = \infty$ from finite-p inequalities. We record the estimate used below.

Endpoint transport–diffusion estimate. Let $0 < \gamma \le 2$, let v be a smooth divergence-free vector field on $\mathbb{T}^d$, and let f solve

$$\partial_t f + v \cdot \nabla f + \nu \Lambda^\gamma f = 0, \qquad f(0) = f_0.$$

Set $V(t) = \int_0^t \|v(\tau)\|_{B^1_{\infty,1}} \, d\tau$. Then

$$\|f\|_{L^\infty_t(B^1_{\infty,1})} + \nu\|f\|_{L^1_t(B^{1+\gamma}_{\infty,1})} \le Ce^{CV(t)}\|f_0\|_{B^1_{\infty,1}}. \tag{2.4}$$

The constant depends on γ and on the fixed Littlewood–Paley decomposition, but not on a Lebesgue exponent p.

This is the standard endpoint Besov transport–diffusion estimate; see Chapter 3 of [1] and the fractional dissipative SQG estimates in [11]. The following commutator estimate plays an important role in this paper. It is the periodic homogeneous version of the generalized Kato–Ponce estimate in [20, Theorem 1.9]; the corresponding fractional Leibniz estimate is recorded in [20, (1.15)]. The periodic form follows by the standard transference argument, and the zero-mean assumption permits the replacement of J^s by Λ^s. The case $s=0$ is immediate.

Lemma 2.3 (Sobolev commutator estimate)**.** Let $s \geq 0$, $p \in (1,\infty)$, $p_1, \cdots, p_4 \in (1,\infty]$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then there exists a constant C such that for any f and g of zero mean

$$\|[\Lambda^s, f\cdot]\nabla g\|_{L^p} \leq C(\|\Lambda^s f\|_{L^{p_1}}\|\nabla g\|_{L^{p_2}} + \|\nabla f\|_{L^{p_3}}\|\Lambda^s g\|_{L^{p_4}}).$$

Lemma 2.4 (Besov commutator estimate)**.** Let $1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $\sigma > -1 - d\min(\frac{1}{p}, \frac{1}{p'})$ and $\operatorname{div}\, \mathrm{u} = 0$. Define $I_j = [u\cdot\nabla, \Delta_j]f$. Then there exists a constant C such that for $\sigma > 0$, we have

$$\left\|(2^{j\sigma}\|I_j\|_{L^p_x})_j\right\|_{l^r(\mathbb{Z})} \leq C(\|\nabla u\|_{L^\infty}\|f\|_{B^\sigma_{p,r}} + \|\nabla u\|_{B^{\sigma-1}_{p,r}}\|\nabla_x f\|_{L^\infty_x}).$$

Moreover, if $\sigma > 1 + \frac{d}{p}$ or $\sigma = 1 + \frac{d}{p}$, $r = 1$, we have

$$\left\|(2^{j\sigma}\|I_j\|_{L^p_x})_j\right\|_{l^r(\mathbb{Z})} \leq C\|\nabla u\|_{B^{\sigma-1}_{p,r}}\|f\|_{B^\sigma_{p,r}};$$

if $\sigma = 1 + \frac{d}{p}$, $r > 1$, we have

$$\left\|(2^{j\sigma}\|I_j\|_{L^p_x})_j\right\|_{l^r(\mathbb{Z})} \leq C\|\nabla u\|_{B^\sigma_{p,r}}\|f\|_{B^\sigma_{p,r}};$$

if $\sigma < 1 + \frac{d}{p}$, we have

$$\left\|(2^{j\sigma}\|I_j\|_{L^p_x})_j\right\|_{l^r(\mathbb{Z})} \leq C\|\nabla u\|_{B^{\frac{d}{p}}_{p,r}\cap L^\infty}\|f\|_{B^\sigma_{p,r}}.$$

We have the following product rules.

Lemma 2.5. For nonlinear terms that satisfy the div-curl structure, i.e. $\operatorname{div} u = 0$, the following $B^0_{\infty,1}$ estimate holds

$$\|u\cdot\nabla v\|_{B^0_{\infty,1}} \leq C\|u\|_{B^0_{\infty,1}}\|v\|_{B^1_{\infty,1}},$$

for some absolute constant $C > 0$.

Remark 2.6. We would like to highlight here that in general the $B^0_{\infty,1}$ space is not a multiplicative algebra, namely, $\|f\cdot g\|_{B^0_{\infty,1}}$ cannot be bounded by $\|f\|_{B^0_{\infty,1}}\cdot\|g\|_{B^0_{\infty,1}}$. However, if the factors satisfy the div–curl structure in Lemma 2.5, the cancellation gives

$$\|f\cdot\nabla g\|_{B^0_{\infty,1}} \lesssim \|f\|_{B^0_{\infty,1}}\|g\|_{B^1_{\infty,1}}.$$

The zero-mean condition alone does not replace this structural assumption.

Lemma 2.7 (Riesz operator $\nabla^\perp\Lambda^{-1}$)**.** Let $1 < p < \infty$ and $f \in L^p(\mathbb{T}^2)$ of zero mean. Then we have

$$\|\nabla^\perp\Lambda^{-1}f\|_{L^p} \leq C_p\|f\|_{L^p}, \tag{2.5}$$

for some constant $C_p > 0$ depending on p.

If $f \in H^s(\mathbb{T}^2)$ is of zero mean and $s > 1$, then

$$\|\nabla^\perp\Lambda^{-1}f\|_{L^\infty} \leq C_s\|f\|_{H^s}. \tag{2.6}$$

Moreover, if $f \in B^0_{\infty,1}(\mathbb{T}^2)$ is of zero mean, then

$$\|\nabla^\perp\Lambda^{-1}f\|_{B^0_{\infty,1}} \leq C\|f\|_{B^0_{\infty,1}}, \tag{2.7}$$

where the constants depend only on the indicated regularity index and on the fixed Littlewood–Paley decomposition.

3. Proof of the Main Theorem

In this section we will prove Theorem 1.1 by showing the local well-posedness and the vanishing viscosity limit of (SQG) in Besov space $B^1_{\infty,1}$. To start with, we first introduce our iteration scheme. To avoid the ambiguity, we shall use the subscript to denote the iteration scheme in this section and the superscript is adopted to denote the numerical discretization.

Proof of Theorem 1.1. Let $\theta_1 = S_1\theta_0$, where S_1 (and S_{k+1} below) is a frequency cut-off operator defined in (2.1). We then define a sequence $\{\theta_{k+1}\}_{k\in\mathbb{N}^+}$ by solving the following linear equations:

$$\begin{cases} \partial_t\theta_{k+1} + u_k\cdot\nabla\theta_{k+1} = -\nu\Lambda^\gamma\theta_{k+1}, \\ u_k = \nabla^\perp\Lambda^{-1}\theta_k, \\ \theta_{k+1}|_{t=0} = S_{k+1}\theta_0. \end{cases} \tag{3.1}$$

Step 1: Uniform bounds.
The endpoint estimate (2.4), with $f=\theta_{k+1}$ and $v=u_k$, yields

$$\begin{aligned} &\|\theta_{k+1}\|_{L^\infty_T(B^1_{\infty,1})} + \nu\|\theta_{k+1}\|_{L^1_T(B^{1+\gamma}_{\infty,1})} \\ &\quad \le C\exp\left(C\int_0^T \|u_k(t)\|_{B^1_{\infty,1}}dt\right)\|S_{k+1}\theta_0\|_{B^1_{\infty,1}}. \end{aligned} \tag{3.2}$$

The Besov boundedness of the Riesz transform gives

$$\int_0^T \|u_k(t)\|_{B^1_{\infty,1}}dt \le CT\|\theta_k\|_{L^\infty_T(B^1_{\infty,1})}. \tag{3.3}$$

Assume inductively that

$$\|\theta_k\|_{L^\infty_T(B^1_{\infty,1})} + \nu\|\theta_k\|_{L^1_T(B^{1+\gamma}_{\infty,1})} \le 4C\|\theta_0\|_{B^1_{\infty,1}}, \tag{3.4}$$

and choose

$$T = \frac{1}{100C^3(1+R)}, \tag{3.5}$$

where $C>1$ is enlarged, if necessary, to dominate the constants in (3.2)–(3.3). The exponential in (3.2) is then bounded by an absolute constant, and hence

$$\|\theta_{k+1}\|_{L^\infty_T(B^1_{\infty,1})} + \nu\|\theta_{k+1}\|_{L^1_T(B^{1+\gamma}_{\infty,1})} \le 4C\|\theta_0\|_{B^1_{\infty,1}}. \tag{3.6}$$

Since $\theta_1 = S_1\theta_0$ and $T\le 1$, the frequency localization of S_1 and Bernstein's inequality give the base case (3.4). Hence induction yields (3.6) for the approximate sequence.

Step 2: Convergence.
For $k\ge 2$, set $\overline{\theta}_k = \theta_{k+1}-\theta_k$ and $\overline{u}_{k-1} = u_k - u_{k-1}$. Subtracting the two corresponding equations in (3.1), we obtain

$$\begin{cases} \partial_t\overline{\theta}_k + u_k\cdot\nabla\overline{\theta}_k = -\nu\Lambda^\gamma\overline{\theta}_k - \overline{u}_{k-1}\cdot\nabla\theta_k, \\ u_k = \nabla^\perp\Lambda^{-1}\theta_k, \\ \overline{\theta}_k|_{t=0} = \Delta_{k+1}\theta_0. \end{cases} \tag{3.7}$$

Here $\overline{u}_{k-1} = \nabla^\perp\Lambda^{-1}\overline{\theta}_{k-1}$. Using Lemmas 2.4 and 2.5, we deduce from (3.7) that

$$\begin{aligned} \|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})} &\le \|\Delta_{k+1}\theta_0\|_{B^0_{\infty,1}} + CT\|u_k\|_{L^\infty_T(B^1_{\infty,1})}\|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})} \\ &\quad + CT\|\theta_k\|_{L^\infty_T(B^1_{\infty,1})}\|\overline{u}_{k-1}\|_{L^\infty_T(B^0_{\infty,1})} \\ &\le C2^{-(k+1)}\|\theta_0\|_{B^1_{\infty,1}} + CT\|\theta_k\|_{L^\infty_T(B^1_{\infty,1})}(\|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})} + \|\overline{\theta}_{k-1}\|_{L^\infty_T(B^0_{\infty,1})}). \end{aligned}$$

The term containing $\overline{\theta}_k$ on the right is absorbed into the left-hand side. More precisely, (3.5) and (3.6) ensure, after decreasing the absolute constant in the definition of T if needed, that

$CT\|\theta_k\|_{L^\infty_T(B^1_{\infty,1})} \le 1/20$. Consequently,

$$\begin{aligned}\|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})} &\le C2^{-(k+1)}\|\theta_0\|_{B^1_{\infty,1}} + CT\|\theta_k\|_{L^\infty_T(B^1_{\infty,1})}\|\overline{\theta}_{k-1}\|_{L^\infty_T(B^0_{\infty,1})} \\ &\le C2^{-(k+1)}\|\theta_0\|_{B^1_{\infty,1}} + \frac{1}{10}\|\overline{\theta}_{k-1}\|_{L^\infty_T(B^0_{\infty,1})}, \qquad k \ge 2.\end{aligned} \tag{3.8}$$

The first difference is treated separately. By the uniform bounds from Step 1,

$$\|\overline{\theta}_1\|_{L^\infty_T(B^0_{\infty,1})} \le \|\theta_2\|_{L^\infty_T(B^0_{\infty,1})} + \|\theta_1\|_{B^0_{\infty,1}} \le C\|\theta_0\|_{B^1_{\infty,1}}.$$

Summing (3.8) from $k = 2$ to m and adding this first difference gives

$$\sum_{k=1}^{m}\|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})} \le C\|\theta_0\|_{B^1_{\infty,1}} + \frac{1}{10}\sum_{k=1}^{m-1}\|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})}.$$

Absorbing the last sum implies that

$$\sum_{k=1}^{\infty}\|\overline{\theta}_k\|_{L^\infty_T(B^0_{\infty,1})} \le C\|\theta_0\|_{B^1_{\infty,1}}.$$

This means that $\{\theta_k\}_{k\ge1}$ is a Cauchy sequence in $L^\infty([0,T]; B^0_{\infty,1})$. Uniform bounds for $\{\theta_k\}_{k\ge1}$ and the Fatou property for Besov spaces [1] ensure that there exists $\theta \in L^\infty([0,T]; B^1_{\infty,1})$ satisfying (SQG) such that

$$\theta_k \to \theta \text{ in } L^\infty([0,T]; B^{s'}_{\infty,1}),$$

where $0 < s' < 1$. Passing to the limit in (3.6) by the Fatou property now proves (1.1). The usual dyadic time-continuity argument for the transport–diffusion equation also gives

$$\theta \in C([0,T]; B^1_{\infty,1}).$$

Moreover, if $\theta_0 \in B^s_{\infty,1}$ for any $s > 1$, the argument of **Step 1** gives a constant $C = C(R, s, \gamma) > 0$ such that

$$\|\theta\|_{L^\infty_T(B^s_{\infty,1})} + \nu\|\theta\|_{L^1_T(B^{s+\gamma}_{\infty,1})} \le C\|\theta_0\|_{B^s_{\infty,1}}. \tag{3.9}$$

This implies that (1.2) is valid.

Step 3: Uniqueness.
Let θ^1 and θ^2 be two solutions of (SQG) with the initial data θ^1_0 and θ^2_0. Then we obtain

$$\begin{cases} \partial_t(\theta^1 - \theta^2) + u^1 \cdot \nabla(\theta^1 - \theta^2) = -\nu\Lambda^\gamma(\theta^1 - \theta^2) - (u^1 - u^2)\cdot\nabla\theta^2, \\ u^1 = \nabla^\perp\Lambda^{-1}\theta^1, \ u^2 = \nabla^\perp\Lambda^{-1}\theta^2, \\ (\theta^1 - \theta^2)|_{t=0} = \theta^1_0 - \theta^2_0. \end{cases} \tag{3.10}$$

Using Lemmas 2.4 and 2.5, we infer from (3.10) that

$$\begin{aligned}&\|\theta^1 - \theta^2\|_{L^\infty_T(B^0_{\infty,1})} \\ &\le \|\theta^1_0 - \theta^2_0\|_{B^0_{\infty,1}} + C\int_0^T \|\theta^1 - \theta^2\|_{B^0_{\infty,1}}\|u^1\|_{B^1_{\infty,1}} + \|u^1 - u^2\|_{B^0_{\infty,1}}\|\theta^2\|_{B^1_{\infty,1}}\,dt \\ &\le \|\theta^1_0 - \theta^2_0\|_{B^0_{\infty,1}} + C\int_0^T \|\theta^1 - \theta^2\|_{B^0_{\infty,1}}(\|\theta^1\|_{B^1_{\infty,1}} + \|\theta^2\|_{B^1_{\infty,1}})dt \\ &\le \|\theta^1_0 - \theta^2_0\|_{B^0_{\infty,1}} + \frac{1}{2}\|\theta^1 - \theta^2\|_{L^\infty_T(B^0_{\infty,1})},\end{aligned}$$

which implies that

$$\|\theta^1 - \theta^2\|_{L^\infty_T(B^0_{\infty,1})} \le C\|\theta^1_0 - \theta^2_0\|_{B^0_{\infty,1}}. \tag{3.11}$$

We thus conclude the uniqueness for (SQG) on $[0,T]$.

Step 4: Continuous dependence and inviscid limit.
Assume that θ^1 and θ^2 are two solutions of (SQG) with the initial data θ^1_0 and θ^2_0. We introduce

a smoothing operator S_N to quantify regularity. Let θ_N^1 and θ_N^2 be two solutions of (SQG) with the initial data $S_N\theta_0^1$ and $S_N\theta_0^2$, respectively. We deduce from (3.11) that

$$\|\theta^1-\theta_N^1\|_{L_T^\infty(B^0_{\infty,1})}\le C\|(Id-S_N)\theta_0^1\|_{B^0_{\infty,1}}\le C2^{-N}\|(Id-S_N)\theta_0^1\|_{B^1_{\infty,1}}. \tag{3.12}$$

and

$$\|\theta^2-\theta_N^2\|_{L_T^\infty(B^0_{\infty,1})}\le C\|(Id-S_N)\theta_0^2\|_{B^0_{\infty,1}}\le C2^{-N}\|(Id-S_N)\theta_0^2\|_{B^1_{\infty,1}} \tag{3.13}$$

Replacing (θ^2,u^2) with (θ_N^1,u_N^1) in (3.10), by Lemma 2.4, we obtain

$$\begin{aligned}\|(\theta^1-\theta_N^1)\|_{L_T^\infty(B^1_{\infty,1})}\le \|(Id-S_N)\theta_0^1\|_{B^1_{\infty,1}}&+C\int_0^T\|\theta^1-\theta_N^1\|_{B^1_{\infty,1}}\|u^1\|_{B^1_{\infty,1}}dt\\&+C\int_0^T\|(u^1-u_N^1)\cdot\nabla\theta_N^1\|_{B^1_{\infty,1}}dt.\end{aligned}$$

Moreover, we have

$$\|(u^1-u_N^1)\cdot\nabla\theta_N^1\|_{B^1_{\infty,1}}\lesssim\|u^1-u_N^1\|_{B^1_{\infty,1}}\|\theta_N^1\|_{B^1_{\infty,1}}+\|u^1-u_N^1\|_{B^0_{\infty,1}}\|\theta_N^1\|_{B^2_{\infty,1}}.$$

According to (3.12), (3.9), and the Bernstein inequalities $\|S_N\theta_0^1\|_{B^2_{\infty,1}}\le C2^N\|\theta_0^1\|_{B^1_{\infty,1}}$, we infer that

$$\|u^1-u_N^1\|_{B^0_{\infty,1}}\|\theta_N^1\|_{B^2_{\infty,1}}\lesssim\|(Id-S_N)\theta_0^1\|_{B^1_{\infty,1}}\|\theta_0^1\|_{B^1_{\infty,1}}.$$

The above inequalities ensure that

$$\|\theta^1-\theta_N^1\|_{L_T^\infty(B^1_{\infty,1})}\le C\|(Id-S_N)\theta_0^1\|_{B^1_{\infty,1}}. \tag{3.14}$$

According to (3.13), similarly, we deduce that

$$\|\theta^2-\theta_N^2\|_{L_T^\infty(B^1_{\infty,1})}\le C\|(Id-S_N)\theta_0^2\|_{B^1_{\infty,1}}. \tag{3.15}$$

According to the interpolation inequality, (3.9) and (3.11), we know

$$\begin{aligned}\|\theta_N^1-\theta_N^2\|_{L_T^\infty(B^1_{\infty,1})}\le&C\|\theta_N^1-\theta_N^2\|^{\frac12}_{L_T^\infty(B^0_{\infty,1})}\|\theta_N^1-\theta_N^2\|^{\frac12}_{L_T^\infty(B^2_{\infty,1})}\\\le&C2^{\frac N2}\|\theta_0^1-\theta_0^2\|^{\frac12}_{B^0_{\infty,1}}(\|\theta_0^1\|_{B^1_{\infty,1}}+\|\theta_0^2\|_{B^1_{\infty,1}})^{\frac12}.\end{aligned} \tag{3.16}$$

Combining (3.14)-(3.16), we conclude that

$$\begin{aligned}\|\theta^1-\theta^2\|_{L_T^\infty(B^1_{\infty,1})}\le C(&\|(Id-S_N)\theta_0^1\|_{B^1_{\infty,1}}+\|(Id-S_N)\theta_0^2\|_{B^1_{\infty,1}})\\&+C2^{\frac N2}\|\theta_0^1-\theta_0^2\|^{\frac12}_{B^0_{\infty,1}}(\|\theta_0^1\|_{B^1_{\infty,1}}+\|\theta_0^2\|_{B^1_{\infty,1}})^{\frac12}.\end{aligned} \tag{3.17}$$

To make the modulus uniform for data near a fixed θ_0, put $\theta_0^1=\theta_0$ and $\theta_0^2=\phi_0$. Since $I-S_N$ is uniformly bounded on $B^1_{\infty,1}$,

$$\|(I-S_N)\phi_0\|_{B^1_{\infty,1}}\le\|(I-S_N)\theta_0\|_{B^1_{\infty,1}}+C\|\phi_0-\theta_0\|_{B^1_{\infty,1}}.$$

We first choose N using only the tail of the fixed datum θ_0 and η, and then choose δ so that all remaining terms in (3.17) are smaller than η. The choices are independent of ν.

Let θ^ν be the solution of (SQG) with the initial data θ_0 and $\nu\ge0$. We also denote θ^0 to be the solution to the inviscid (SQG) equations. Then we get

$$\begin{cases}\partial_t(\theta^\nu-\theta^0)+u^\nu\cdot\nabla(\theta^\nu-\theta^0)=-\nu\Lambda^\gamma\theta^\nu-(u^\nu-u^0)\cdot\nabla\theta^0,\\ u^\nu=\nabla^\perp\Lambda^{-1}\theta^\nu,\\ (\theta^\nu-\theta^0)|_{t=0}=0.\end{cases} \tag{3.18}$$

Set $w=\theta^\nu-\theta^0$ and $M=\sup_{\mu\in[0,1]}\|\theta^\mu\|_{L_T^\infty(B^1_{\infty,1})}$. By (3.6), M is finite and depends only on the radius of the initial-data ball. Lemmas 2.4 and 2.5, applied to (3.18), give the common starting estimate

$$\|w\|_{L_T^\infty(B^0_{\infty,1})}\le C\nu\int_0^T\|\theta^\nu\|_{B^\gamma_{\infty,1}}dt+C\int_0^T\|w\|_{B^0_{\infty,1}}(\|\theta^\nu\|_{B^1_{\infty,1}}+\|\theta^0\|_{B^1_{\infty,1}})dt. \tag{3.19}$$

For $\gamma \in (0,1]$, the monotonicity of the Besov scale and the uniform $B^1_{\infty,1}$ bound imply

$$\nu \int_0^T \|\theta^\nu\|_{B^\gamma_{\infty,1}} dt \le \nu T \|\theta^\nu\|_{L^\infty_T(B^1_{\infty,1})} \le C\nu.$$

For $\gamma \in (1,2]$, real interpolation between $B^1_{\infty,1}$ and $B^{1+\gamma}_{\infty,1}$ yields, pointwise in time,

$$\|\theta^\nu\|_{B^\gamma_{\infty,1}} \le C \|\theta^\nu\|^{1/\gamma}_{B^1_{\infty,1}} \|\theta^\nu\|^{1-1/\gamma}_{B^{1+\gamma}_{\infty,1}}.$$

Hölder's inequality in time, followed by (3.6), therefore gives

$$\begin{aligned} \nu \int_0^T \|\theta^\nu\|_{B^\gamma_{\infty,1}} dt &\le C\nu T^{1/\gamma} M^{1/\gamma} \|\theta^\nu\|^{1-1/\gamma}_{L^1_T(B^{1+\gamma}_{\infty,1})} \\ &= C\nu^{1/\gamma} T^{1/\gamma} M^{1/\gamma} \big(\nu \|\theta^\nu\|_{L^1_T(B^{1+\gamma}_{\infty,1})}\big)^{1-1/\gamma} \le C\nu^{1/\gamma}. \end{aligned}$$

Finally, the second term on the right-hand side of (3.19) is at most $CTM\|w\|_{L^\infty_T(B^0_{\infty,1})}$, and our choice of T makes $CTM \le 1/2$. Absorbing that term proves

$$\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})} \le C\nu^{\min\{1,\frac{1}{\gamma}\}}. \tag{3.20}$$

Let θ^ν_N and θ^0_N be the viscous and inviscid solutions, respectively, both with initial data $S_N\theta_0$. Applying (3.15) once with viscosity ν and once with viscosity zero gives

$$\|\theta^\nu - \theta^\nu_N\|_{L^\infty_T(B^1_{\infty,1})} + \|\theta^0 - \theta^0_N\|_{L^\infty_T(B^1_{\infty,1})} \le C\|(Id - S_N)\theta_0\|_{B^1_{\infty,1}}. \tag{3.21}$$

For completeness, persistence of higher regularity and Bernstein's inequality give

$$\|\theta^\nu_N\|_{L^\infty_T(B^{1+\gamma}_{\infty,1})} \le C2^{\gamma N}\|\theta_0\|_{B^1_{\infty,1}}, \qquad \|\theta^0_N\|_{L^\infty_T(B^2_{\infty,1})} \le C2^N\|\theta_0\|_{B^1_{\infty,1}}.$$

Moreover, (3.20), applied to the common truncated initial datum, has the same constant because $\|S_N\theta_0\|_{B^1_{\infty,1}} \le C\|\theta_0\|_{B^1_{\infty,1}}$. Using Lemmas 2.4 and 2.5, we therefore infer that

$$\begin{aligned} &\|\theta^\nu_N - \theta^0_N\|_{L^\infty_T(B^1_{\infty,1})} \\ \le& C\nu \int_0^T \|\Lambda^\gamma \theta^\nu_N\|_{B^1_{\infty,1}} dt + C\int_0^T \|u^\nu_N - u^0_N\|_{B^0_{\infty,1}} \|\theta^0_N\|_{B^2_{\infty,1}} dt \\ &+ C\int_0^T \|\theta^\nu_N - \theta^0_N\|_{B^1_{\infty,1}} (\|\theta^\nu_N\|_{B^1_{\infty,1}} + \|\theta^0_N\|_{B^1_{\infty,1}}) dt \\ \le& C\nu 2^{\gamma N} + C\nu^{\min\{1,\frac{1}{\gamma}\}} 2^N + \frac{1}{2}\|\theta^\nu_N - \theta^0_N\|_{L^\infty_T(B^1_{\infty,1})}, \end{aligned}$$

where the last term is absorbed into the left-hand side. The triangle inequality and (3.21) now ensure that

$$\|\theta^\nu - \theta^0\|_{L^\infty_T(B^1_{\infty,1})} \lesssim \nu 2^{\gamma N} + \nu^{\min\{1,\frac{1}{\gamma}\}} 2^N + \|(Id - S_N)\theta_0\|_{B^1_{\infty,1}}. \tag{3.22}$$

To make the last passage explicit, choose $N = N(\nu)$ so that $2^{\gamma N} \sim \nu^{-\epsilon}$ with $0 < \epsilon < \min\{1,\gamma\}$. Then the first two terms on the right-hand side of (3.22) tend to zero, while $N(\nu) \to \infty$ and hence $\|(Id - S_{N(\nu)})\theta_0\|_{B^1_{\infty,1}} \to 0$. This proves (1.5). We thus complete the proof of Theorem 1.1 and Remark 1.4. □

We now sketch the proof of Proposition 1.5.

Proof of Proposition 1.5. Similarly, we infer that

$$\|\theta^\nu\|_{L^\infty_T(B^\alpha_{\infty,1})} + \nu\|\theta^\nu\|_{L^1_T(B^{\alpha+\gamma}_{\infty,1})} \le 4C\|\theta_0\|_{B^\alpha_{\infty,1}}. \tag{3.23}$$

where $\alpha \in (1,2]$. Using Lemma 2.4, for $\gamma \in (0,\alpha]$, we infer from (3.18) and (3.23) that

$$\begin{aligned}
&\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})}\\
\le& C\nu \int_0^T \|\Lambda^\gamma \theta^\nu\|_{B^0_{\infty,1}} dt + C\int_0^T \|\theta^\nu - \theta^0\|_{B^0_{\infty,1}} \|u^\nu\|_{B^1_{\infty,1}} + \|u^\nu - u^0\|_{B^0_{\infty,1}} \|\theta^0\|_{B^1_{\infty,1}} dt\\
\le& C\nu + C\int_0^T \|\theta^\nu - \theta^0\|_{B^0_{\infty,1}} (\|\theta^\nu\|_{B^1_{\infty,1}} + \|\theta^0\|_{B^1_{\infty,1}}) dt\\
\le& C\nu + \frac{1}{2}\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})}.
\end{aligned}$$

For $\gamma \in (\alpha, 2]$, we infer from (3.18) and (3.23) that

$$\begin{aligned}
&\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})}\\
\le& C\nu \int_0^T \|\Lambda^\gamma \theta^\nu\|_{B^0_{\infty,1}} dt + C\int_0^T \|\theta^\nu - \theta^0\|_{B^0_{\infty,1}} \|u^\nu\|_{B^1_{\infty,1}} + \|u^\nu - u^0\|_{B^0_{\infty,1}} \|\theta^0\|_{B^1_{\infty,1}} dt\\
\le& C\nu \int_0^T \|\theta^\nu\|^{\frac{\alpha}{\gamma}}_{B^\alpha_{\infty,1}} \|\theta^\nu\|^{1-\frac{\alpha}{\gamma}}_{B^{\alpha+\gamma}_{\infty,1}} dt + C\int_0^T \|\theta^\nu - \theta^0\|_{B^0_{\infty,1}} (\|\theta^\nu\|_{B^1_{\infty,1}} + \|\theta^0\|_{B^1_{\infty,1}}) dt\\
\le& C\nu^{\frac{\alpha}{\gamma}} (\nu \|\theta^\nu\|_{L^1_T(B^{\alpha+\gamma}_{\infty,1})})^{1-\frac{\alpha}{\gamma}} + \frac{1}{2}\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})}\\
\le& C\nu^{\frac{\alpha}{\gamma}} + \frac{1}{2}\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})}.
\end{aligned}$$

Then we conclude from the above two inequalities that

$$\|\theta^\nu - \theta^0\|_{L^\infty_T(B^0_{\infty,1})} \le C\nu^{\min\{1,\frac{\alpha}{\gamma}\}}.$$

□

Remark 3.1 (Overcoming Derivative Loss in Sobolev Spaces)**.** A fundamental obstacle in establishing the inviscid limit in the exact space H^s is that standard energy estimates for the nonlinear advection term suffer from a severe derivative loss. For the SQG equations, this difficulty is drastically amplified by the strongly singular Riesz transform in the velocity $u^\nu = \nabla^\perp \Lambda^{-1}\theta^\nu$. The Bona-Smith type frequency truncation employed in Step 2 of the proof precisely resolves this dilemma: it optimally balances the high-norm growth of the truncated solution with the low-norm strong convergence, enabling the continuous dependence and inviscid limit without artificially inflating the regularity requirements.

4. Inviscid limit in Sobolev spaces

In this subsection, we establish the quantitative vanishing viscosity limit in the standard Sobolev framework and therefore prove Proposition 1.7. By utilizing the frequency cut-off operator S_N, we construct a quantitative method to estimate the vanishing rate. We first show the uniform boundedness in H^s.

Lemma 4.1 (Uniform H^s bounds)**.** Let $s > 2$ and $\gamma \in (0,2]$. Assume that the initial data $\theta_0 \in H^s(\mathbb{T}^2)$ has zero mean. Then there exists a time $T > 0$, depending only on s and $\|\theta_0\|_{H^s}$ and independent of $\nu \in [0,1]$, such that the viscous SQG equations (SQG) admit a unique solution $\theta^\nu \in C([0,T]; H^s(\mathbb{T}^2))$. Moreover, the solution satisfies the uniform bound:

$$\sup_{t\in[0,T]} \|\theta^\nu(t)\|_{H^s} \le 2\|\theta_0\|_{H^s}, \quad \forall \nu \in [0,1]. \tag{4.1}$$

Proof. We perform classical energy estimates in H^s. Applying the fractional Laplacian Λ^s to the (SQG) equations and taking the L^2 inner product with $\Lambda^s\theta^\nu$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta^\nu\|^2_{H^s} + \nu\|\Lambda^{s+\frac{\gamma}{2}}\theta^\nu\|^2_{L^2} = -\int_{\mathbb{T}^2} \Lambda^s(u^\nu\cdot\nabla\theta^\nu)\Lambda^s\theta^\nu\, dx. \tag{4.2}$$

Since $\operatorname{div} u^\nu = 0$, the nonlinear term can be rewritten entirely as a commutator:

$$-\int_{\mathbb{T}^2} \Lambda^s(u^\nu\cdot\nabla\theta^\nu)\Lambda^s\theta^\nu\, dx = -\int_{\mathbb{T}^2} [\Lambda^s, u^\nu\cdot\nabla]\theta^\nu\Lambda^s\theta^\nu\, dx.$$

Applying the Cauchy-Schwarz inequality and Lemma 2.3, we can bound this term as follows:

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2}[\Lambda^s, u^\nu\cdot\nabla]\theta^\nu\Lambda^s\theta^\nu\,dx\right| \\
\le&\|[\Lambda^s, u^\nu\cdot\nabla]\theta^\nu\|_{L^2}\|\theta^\nu\|_{H^s} \\
\lesssim&\left(\|\nabla u^\nu\|_{L^\infty}\|\Lambda^{s-1}\nabla\theta^\nu\|_{L^2}+\|\nabla\theta^\nu\|_{L^\infty}\|\Lambda^s u^\nu\|_{L^2}\right)\|\theta^\nu\|_{H^s} \\
\lesssim&\left(\|\nabla u^\nu\|_{L^\infty}+\|\nabla\theta^\nu\|_{L^\infty}\right)\|\theta^\nu\|_{H^s}^2,
\end{aligned}\tag{4.3}$$

where we used the fact that the Riesz transform is bounded in H^s, which implies $\|u^\nu\|_{H^s}=\|\nabla^\perp\Lambda^{-1}\theta^\nu\|_{H^s}\lesssim\|\theta^\nu\|_{H^s}$.

Since we assume $s>2$, the Sobolev embedding $H^{s-1}(\mathbb{T}^2)\hookrightarrow L^\infty(\mathbb{T}^2)$ ensures that

$$\|\nabla\theta^\nu\|_{L^\infty}\lesssim\|\theta^\nu\|_{H^s},\quad\text{and}\quad\|\nabla u^\nu\|_{L^\infty}\lesssim\|u^\nu\|_{H^s}\lesssim\|\theta^\nu\|_{H^s}.$$

Substituting these embedding bounds back into (4.3) and dropping the non-negative dissipation term $\nu\|\Lambda^{s+\frac{\gamma}{2}}\theta^\nu\|_{L^2}^2$ on the left-hand side of (4.2), we deduce the differential inequality:

$$\frac{d}{dt}\|\theta^\nu\|_{H^s}^2\le C_0\|\theta^\nu\|_{H^s}^3\implies\frac{d}{dt}\|\theta^\nu\|_{H^s}\le C_1\|\theta^\nu\|_{H^s}^2,$$

where $C_1=C_1(s)>0$ is independent of ν. Solving this ordinary differential inequality yields

$$\|\theta^\nu(t)\|_{H^s}\le\frac{\|\theta_0\|_{H^s}}{1-C_1t\|\theta_0\|_{H^s}}.$$

By choosing $T=\frac{1}{2C_1\|\theta_0\|_{H^s}}$, we guarantee that the denominator is at least $1/2$ for all $t\in[0,T]$. Consequently, we obtain the uniform bound $\sup_{t\in[0,T]}\|\theta^\nu(t)\|_{H^s}\le 2\|\theta_0\|_{H^s}$. The existence and uniqueness then follow from standard approximation techniques (e.g., Galerkin methods), and the lifespan T is clearly independent of ν. This completes the proof. □

Proof of Proposition 1.7. Let $w=\theta^\nu-\theta^0$ denote the error between the viscous and inviscid solutions. It satisfies the following error equation:

$$\begin{cases}\partial_t w+u^\nu\cdot\nabla w=-\nu\Lambda^\gamma\theta^\nu-(u^\nu-u^0)\cdot\nabla\theta^0,\\ u^\nu-u^0=\nabla^\perp\Lambda^{-1}w,\\ w|_{t=0}=0.\end{cases}\tag{4.4}$$

Step 1: Low-norm estimate without derivative loss.
We consider L^2 case first, then the energy estimate gives

$$\frac12\frac{d}{dt}\|w\|_{L^2}^2+\nu(\Lambda^\gamma\theta^\nu,w)=-\left((u^\nu-u^0)\cdot\nabla\theta^0,w\right)$$

by the incompressibility. Then

$$\begin{aligned}&|-\nu(\Lambda^\gamma\theta^\nu,w)|\le\nu\|\theta^\nu\|_{H^s}\|w\|_{L^2},\\ &\|(u^\nu-u^0)\cdot\nabla\theta^0\|_{L^2}\le\|u^\nu-u^0\|_{L^2}\|\nabla\theta^0\|_{L^\infty}\le C\|w\|_{L^2}\|\theta^0\|_{H^s}.\end{aligned}$$

Noting the uniform H^s bounds for θ^ν and θ^0, we deduce that

$$\frac{d}{dt}\|w\|_{L^2}\le C\nu+C\|w\|_{L^2},$$

and by Grönwall's inequality we arrive at $\|w\|_{L^\infty_TL^2}\le C\nu$ which shows (1.6).

We first establish the estimate with a fixed loss of one derivative, (1.7). Applying the preceding energy argument at the level $r=s-1$, using Lemma 2.3, the fractional Leibniz estimate, and the uniform H^s bounds, gives

$$\frac{d}{dt}\|w\|_{H^{s-1}}\le C\|w\|_{H^{s-1}}+C\nu\|\theta^\nu\|_{H^{s-1+\gamma}}.\tag{4.5}$$

If $0<\gamma\le 1$, then $\|\theta^\nu\|_{H^{s-1+\gamma}}\le C\|\theta^\nu\|_{H^s}$, and Grönwall's inequality yields

$$\|w\|_{L^\infty_T(H^{s-1})}\le C\nu.$$

If $1 < \gamma \leq 2$, retaining the dissipative term in the H^s energy estimate (4.2) and integrating in time gives

$$\nu \int_0^T \|\theta^\nu(t)\|_{H^{s+\gamma/2}}^2 \, dt \leq C. \tag{4.6}$$

Set $a = 2(\gamma-1)/\gamma \in (0,1]$. Interpolation and Hölder's inequality in time imply

$$\begin{aligned}\nu \int_0^T \|\theta^\nu\|_{H^{s-1+\gamma}} \, dt &\leq C\nu \int_0^T \|\theta^\nu\|_{H^s}^{1-a} \|\theta^\nu\|_{H^{s+\gamma/2}}^a \, dt \\ &\leq C\nu \left(\int_0^T \|\theta^\nu\|_{H^{s+\gamma/2}}^2 \, dt \right)^{a/2} \leq C\nu^{1-a/2} = C\nu^{1/\gamma}.\end{aligned}$$

Integrating (4.5) and applying Grönwall's inequality therefore prove

$$\|w\|_{L_T^\infty(H^{s-1})} \leq C\nu^{\min\{1,1/\gamma\}}, \qquad 0 < \gamma \leq 2, \tag{4.7}$$

which is the Sobolev estimate directly corresponding to the $B^1_{\infty,1} \to B^0_{\infty,1}$ rate.

We next prove (1.8). First suppose that $1 \leq \gamma \leq 2$ and put $r = s - \gamma$. Then $0 < r \leq s-1$. Applying Λ^r to (4.4), taking the L^2 inner product with $\Lambda^r w$, and using $\operatorname{div} u^\nu = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{H^r}^2 = &-\nu(\Lambda^s\theta^\nu, \Lambda^r w) - (\Lambda^r((u^\nu - u^0)\cdot\nabla\theta^0), \Lambda^r w) \\ &- ([\Lambda^r, u^\nu\cdot\nabla]w, \Lambda^r w).\end{aligned} \tag{4.8}$$

Lemma 2.3, the fractional Leibniz estimate cited above, the Riesz-transform bound, and the uniform H^s estimate give

$$\begin{aligned}\|(u^\nu - u^0)\cdot\nabla\theta^0\|_{H^r} &\leq C\|w\|_{H^r}\|\theta^0\|_{H^s}, \\ \big|([\Lambda^r, u^\nu\cdot\nabla]w, \Lambda^r w)\big| &\leq C\|\theta^\nu\|_{H^s}\|w\|_{H^r}^2, \\ \nu\big|(\Lambda^s\theta^\nu, \Lambda^r w)\big| &\leq C\nu\|w\|_{H^r}.\end{aligned}$$

Consequently,

$$\frac{d}{dt}\|w\|_{H^r} \leq C\nu + C\|w\|_{H^r},$$

and Grönwall's inequality yields

$$\|w\|_{L_T^\infty(H^{s-\gamma})} \leq C\nu, \qquad 1 \leq \gamma \leq 2. \tag{4.9}$$

Now let $0 < \gamma < 1$. In this case (4.7) gives $\|w\|_{L_T^\infty(H^{s-1})} \leq C\nu$. On the other hand, the uniform H^s bounds for the two solutions imply $\|w\|_{L_T^\infty(H^s)} \leq C$. Interpolating between H^{s-1} and H^s gives

$$\|w\|_{H^{s-\gamma}} \leq C\|w\|_{H^{s-1}}^\gamma \|w\|_{H^s}^{1-\gamma} \leq C\nu^\gamma,$$

which, together with (4.9), proves (1.8).

Step 2: Continuous dependence.
To estimate the critical H^s norm, let θ_N^ν be the solution with initial datum $S_N\theta_0$ and put $v = \theta^\nu - \theta_N^\nu$. Then

$$\partial_t v + u^\nu\cdot\nabla v = -\nu\Lambda^\gamma v - (u^\nu - u_N^\nu)\cdot\nabla\theta_N^\nu, \qquad v(0) = (Id - S_N)\theta_0.$$

We first work one derivative lower. Lemma 2.3, the fractional Leibniz estimate cited above, the non-negativity of the dissipative term, and the uniform H^s bounds give

$$\frac{d}{dt}\|v\|_{H^{s-1}} \leq C\|v\|_{H^{s-1}},$$

and hence

$$\|v\|_{L_T^\infty(H^{s-1})} \leq C\|(Id - S_N)\theta_0\|_{H^{s-1}} \leq C2^{-N}\|(Id - S_N)\theta_0\|_{H^s}. \tag{4.10}$$

Because $s - 1 > 1$ in two dimensions, $H^{s-1} \hookrightarrow L^\infty$. Persistence of regularity for the truncated solution and Bernstein's inequality also give

$$\|\theta_N^\nu\|_{L_T^\infty(H^{s+1})} \leq C\|S_N\theta_0\|_{H^{s+1}} \leq C2^N\|\theta_0\|_{H^s}.$$

Thus the apparently dangerous factor is controlled by the actual high-frequency tail:

$$\begin{aligned}\|u^\nu - u_N^\nu\|_{L^\infty}\|\theta_N^\nu\|_{H^{s+1}} &\le C\|v\|_{H^{s-1}}\|\theta_N^\nu\|_{H^{s+1}} \\ &\le C\|(Id - S_N)\theta_0\|_{H^s}\|\theta_0\|_{H^s}. \end{aligned} \tag{4.11}$$

The H^s energy estimate for v, followed by (4.11) and Grönwall's inequality, now yields

$$\|\theta^\nu - \theta_N^\nu\|_{L_T^\infty(H^s)} \le C\|(Id - S_N)\theta_0\|_{H^s}. \tag{4.12}$$

The same argument applies when $\nu = 0$ and gives the corresponding bound for $\theta^0 - \theta_N^0$. Notice that no comparison of the form $2^{-cN} \lesssim \|(Id - S_N)\theta_0\|_{H^s}$ is used; such a comparison is false for general initial data.

Step 3: High-norm truncation error and synthesis.
Let $w_N = \theta_N^\nu - \theta_N^0$. A preliminary H^{s-1} estimate, using

$$\|\Lambda^\gamma \theta_N^\nu\|_{H^{s-1}} \le C2^{(\gamma-1)_+N}\|\theta_0\|_{H^s},$$

gives

$$\|w_N\|_{L_T^\infty(H^{s-1})} \le C\nu 2^{(\gamma-1)_+N}. \tag{4.13}$$

At the H^s level the same Kato–Ponce and fractional Leibniz estimates give

$$\frac{d}{dt}\|w_N\|_{H^s} \le C\|w_N\|_{H^s} + C\|w_N\|_{H^{s-1}}\|\theta_N^0\|_{H^{s+1}} + C\nu\|\theta_N^\nu\|_{H^{s+\gamma}}.$$

Persistence of higher regularity, applied to the truncated initial data, implies

$$\|\theta_N^0\|_{L_T^\infty(H^{s+1})} \le C2^N, \qquad \|\theta_N^\nu\|_{L_T^\infty(H^{s+\gamma})} \le C2^{\gamma N}.$$

Combining these bounds with (4.13) and using Grönwall's inequality, we obtain

$$\|\theta_N^\nu - \theta_N^0\|_{L_T^\infty(H^s)} \le C\nu 2^{\max\{1,\gamma\}N}. \tag{4.14}$$

Consequently,

$$\|\theta^\nu - \theta^0\|_{L_T^\infty(H^s)} \le C\left(\|(Id - S_N)\theta_0\|_{H^s} + \nu 2^{\max\{1,\gamma\}N}\right).$$

First choose N so that the high-frequency tail is small and then let $\nu \to 0$. This proves (1.9) and completes the proof of Proposition 1.7.

□

5. Concluding Remarks on Rough Data

In the study of fluid dynamics, understanding the inviscid limit of solutions with non-smooth initial data is of profound physical importance. For instance, the optimal convergence rate for 2D and 3D vortex patch initial data in the Navier-Stokes to Euler transition has been a subject of intense mathematical interest. The SQG equations exhibit significantly more singular behavior than the Euler equations, making the corresponding patch problem extremely challenging.

The rigorous analytical framework established in this paper, which operates at the borderline regularity space $B^1_{\infty,1}$ without assuming higher-order Sobolev smoothness ($s > 2$), avoids the catastrophic derivative loss induced by the singular Riesz transforms. This methodology paves the way for future investigations into the vanishing viscosity limit of SQG equations with rough data, including generalized temperature patches and piecewise smooth initial fronts.

Acknowledgement

X. Cheng is partially supported by NSFC (Grants No. 12401270 and No. 42450192), Natural Science Foundation of Shanghai (Grant No. 24ZR1404200) and Shanghai Magnolia Talent Plan Pujiang Project (Grant No. 24PJA007). S. Wang is partially supported by the China National Postdoctoral Program for Innovative Talents (Grant No. 20250061) and China Postdoctoral Science Foundation (Grant No. 2024M760058).

References

[1] H. Bahouri, J. Y. Chemin and R. Danchin. Fourier analysis and nonlinear partial differential equations, *Springer, Heidelberg*, 2011.
[2] J. T. Beale, T. Kato and A. Majda. Remarks on the Breakdown of Smooth Solutions for the 3D Euler Equations. *Comm. Math. Phys.*, 94(1):61–66 1984.
[3] J. Bourgain, D. Li. Strong ill-posedness of the Incompressible Euler Equation in Borderline Sobolev Spaces. *Invent. Math.*, 201(1):97–157, 2015.
[4] J. Bourgain, D. Li. Strong illposedness of the incompressible Euler equation in integer C^m spaces. *Geometric and Functional Analysis*, 25(1): 1-86, 2015.
[5] T. Buckmaster, S. Shkoller and V. Vicol. Nonuniqueness of weak solutions to the SQG equation. *Commun. Pure Appl. Math.*, 72(9): 1809–1874, 2019.
[6] X. Cheng, H. Kwon and D. Li. Non-uniqueness of stationary weak solutions to the surface quasi-geostrophic equations. *Comm. Math. Phys.*, 388 (3): 1281-1295, 2021.
[7] X. Cheng, Z. Luo, Z. Yang and C. Yuan. Global well-posedness and uniform-in-time vanishing damping limit for the inviscid Oldroyd-B model, preprint, arXiv:2410.09340.
[8] P. Constantin, M. Ignatova and Q. H. Nguyen. Global regularity for critical SQG in bounded domains. *Communications on Pure and Applied Mathematics*, 2024.
[9] P. Constantin, M. Ignatova and Q. H. Nguyen. Inviscid Limit for SQG in Bounded Domains. *SIAM Journal on Mathematical Analysis*, 50(6): 6196-6207, 2018.
[10] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2): 377–407, 2013.
[11] H. Dong and D. Li. On the 2D critical and supercritical dissipative quasi-geostrophic equation in Besov spaces. *J. Differential Equations*, 248(11): 2684-2702, 2010.
[12] V. Giri, H. Kwon and M. Novack. The L^3-based strong Onsager theorem. *Annals of Mathematics.*, to appear.
[13] Z. Guo, J. Li and Z. Yin. Local Well-posedness of the Incompressible Euler Equations in $B^1_{\infty,1}$ and the Inviscid Limit of the Navier-Stokes Equations. *J. Funct. Anal.*, 276(9): 2821–2830, 2019.
[14] I. M. Held, R. T. Pierrehumbert, S. T. Garner and K. L. Swanson. Surface quasi-geostrophic dynamics. *J. Fluid Mech.*, 282:1-20, 1995.
[15] T. Hmidi and S. Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. for Rational Mech. and Analysis*, 189(2):283–300, 2008.
[16] P. Isett. A proof of Onsager's conjecture, *Ann. of Math.*, 188(3): 871–963, 2018.
[17] I.J. Jeong and J. Kim. Strong ill-posedness for SQG in critical Sobolev spaces. *Analysis & PDE*, 17(1): 133-170, 2024.
[18] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure App. Math.*, 41: 891–907, 1988.
[19] J. Leray. Sur Le Mouvement Dun Liquide Visqueux Emplissant Lespace. *Acta Math.*, 63(1):193–248, 1934.
[20] D. Li. On Kato–Ponce and fractional Leibniz. *Rev. Mat. Iberoam.*, 35(1): 23–100, 2019.
[21] D. Li. Optimal gevrey regularity for supercritical quasi-geostrophic equations. *Communications in Mathematical Physics*, 405(2): 30, 2024.
[22] F. Marchand. Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces L^p or $\dot{H}^{-\frac{1}{2}}$, *Comm. Math. Phys.*, 277(1): 45–67, 2008.
[23] N. Masmoudi. Remarks about the inviscid limit of the Navier-Stokes system, *Commun. Math. Phys.*, 270(3): 777–788, 2007.
[24] J. Pedlosky. *Geophysical fluid dynamics.* Springer, New York, 1982.
[25] S. G. Resnick. Dynamical problems in non-linear advective partial differential equations, Ph.D. thesis, University of Chicago, 1995, available at https://www.proquest.com/docview/304242616.
[26] V. I. Yudovich. Non-stationary Flows of an Ideal Incompressible Fluid. *Z. Vycisl. Mat i Mat. Fiz.*, 3:1032–1066, 1963.

X. Cheng, Research Institute of Intelligent Complex Systems, Fudan University, Shanghai, P.R. China
Email address: `xycheng@fudan.edu.cn`

Z. Luo, School of Science, Shenzhen Campus of Sun Yat-sen University, Shenzhen, P.R. China
Email address: `luozhn7@mail.sysu.edu.cn`

S. Wang, School of Mathematical Sciences, Peking University, Beijing, P.R. China
Email address: `everest_sheng@pku.edu.cn`