Optimal Sobolev Estimates on Parabolic Equation with Critical Rough Coefficients

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31st Oct., 2026

Today Is A Super Nice Day!



Cauchy Problem of Parabolic Equations

$(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$:

- Parabolic equation with non–divergence form:

$$\begin{cases} \partial_t u - a^{ij}(t,x)\partial^2_{ij} u + b^i(t,x)\partial_i u + c(t,x)u = f(t,x), \\ u|_{t=0} = u_0(x). \end{cases} \tag{ND}$$

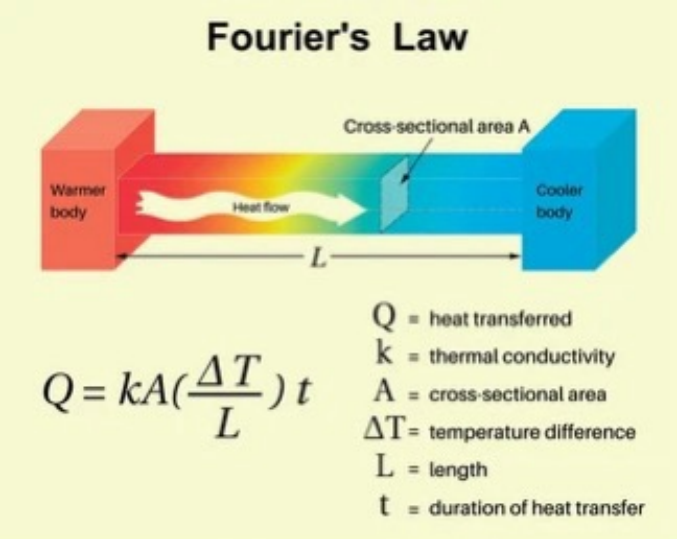
- Parabolic equation with divergence form[1]:

$$\begin{cases} \partial_t u - \partial_i(a^{ij}(t,x)\partial_j u) + b^i(t,x)\partial_i u + c(t,x)u = f(t,x), \\ u|_{t=0} = u_0(x). \end{cases} \tag{D}$$

- $\{a^{ij}(t,x)\}_{d\times d}$: diffusion coefficients (strictly positive-definite).

[1] The divergence parabolic operator is defined via distribution.

Background: Variable Thermal Conductivity



- Fourier heat law:

$$dQ = -k(t,x)\frac{\partial u}{\partial \vec{n}} dSdt.$$

$u(t,x)$: Temperature; $k(t,x)$: Thermal conductivity.

- Heat equation with variable coefficient:

$$\partial_t u - \nabla_x \cdot (k(t,x)\nabla_x u) = 0.$$

Background: Composition Material

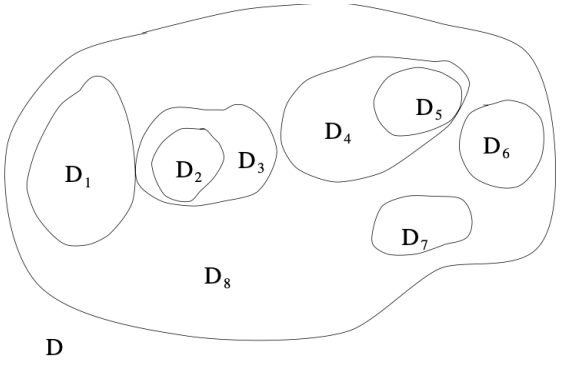
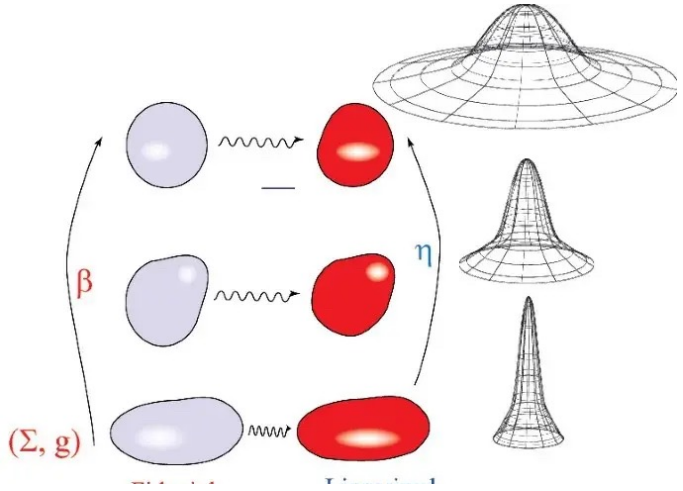


Figure: Composition Material

- D_m: Fibers; $D \setminus (\cup D_m)$: Matrix surrounding fibers.
- Different shear moduli between fibers and matrix lead to discontinuity of the coefficient in the shear equation!

Background: Geometry PDEs



- Equation of Ricci flow:

$$\partial_t g_{ij} = -2R_{ij}(\boldsymbol{g}).$$

- Variable coefficient/Non-linear parabolic equations.

Credit-Rating Migration: The Asset Model

- A zero-coupon bond has face value F and maturity T. Default occurs only at T, with payment $g(S_T) = \min\{S_T, F\}$.
- r: the risk free interest rate; S_t: firm's asset value.

A prescribed rating boundary $b(t) > 0$ determines the volatility:

$$\sigma(t,s) = \begin{cases} \sigma_H, & s \geq b(t) \quad \text{(high rating)}, \\ \sigma_L, & s < b(t) \quad \text{(low rating)}, \end{cases} \qquad 0 < \sigma_H < \sigma_L.$$

Under the risk-neutral measure $\mathbb{Q}$,

$$\boxed{dS_t = rS_t\,dt + \sigma(t,S_t)S_t\,dW_t^{\mathbb{Q}}.}$$

Bond Pricing: The Risk-Neutral Expectation

The bond value is the risk-neutral value of its terminal payment:

$$\boxed{\Phi(t,s) = \mathbb{E}^{\mathbb{Q}}_{t,s}\left[e^{-r(T-t)}\min\{S_T, F\}\right].}$$

$\mathbb{E}^{\mathbb{Q}}_{t,s}$ uses the asset SDE started at $S_t = s$.

Replication is the financial justification

A self-financing portfolio $X_u = \Delta_u S_u + \beta_u B_u$ satisfies

$$dX_u = \Delta_u\, dS_u + r(X_u - \Delta_u S_u)\, du,$$
$$d(e^{-ru}X_u) = e^{-ru}\Delta_u \sigma(u, S_u) S_u\, dW^{\mathbb{Q}}_u.$$

Brownian martingale representation gives a replicating portfolio for the bounded payoff, with $X_T = g(S_T)$ and a true discounted martingale. Hence

$$X_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(S_T) \mid \mathcal{F}_t] = \Phi(t, S_t).$$

Credit-Rating Migration: From SDE to PDE

Inside either rating region, let

$$\mathcal{L}_t = rs\partial_s + \tfrac{1}{2}\sigma^2(t,s)s^2\partial_{ss}.$$

Itô's formula gives discounted drift $e^{-r(u-t)}(\partial_t + \mathcal{L}_u - r)\Phi$. The martingale property therefore yields

$$\boxed{\begin{aligned}\Phi_t + \tfrac{1}{2}\sigma^2(t,s)s^2\Phi_{ss} + rs\Phi_s - r\Phi = 0,\\ \Phi(T,s) = \min\{s,F\}.\end{aligned}}$$

Set $\tau = T - t$, $x = \log s$, and $v(\tau,x) = e^{r\tau}\Phi(T-\tau,e^x)$. Then

$$\boxed{v_\tau - av_{xx} + (a-r)v_x = 0,} \qquad a = \tfrac{1}{2}\sigma^2(T-\tau,e^x).$$

$a(\tau,x)$ jumps across $x = \log b(T-\tau)$: a non-divergence-form parabolic principal part with a discontinuous coefficient.

Settings

- To illustrate the key estimates, consider the following 1-D parabolic equation without lower-order terms

$$\begin{cases} \partial_t u - a(t,x)\partial_x^2 u = f(t,x), \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u|_{t=0} = u_0(x). \end{cases} \tag{1}$$

- The parabolic scaling:

$$u(t,x) \to u(\lambda^2 t, \lambda x), \ a(t,x) \to a(\lambda^2 t, \lambda x).$$

  Critical regularity space $\mathbb{X}$ of the coefficient:

$$\|a\|_{\mathbb{X}} \sim \|a_\lambda\|_{\mathbb{X}}, \ \forall \lambda > 0.$$

- For instance: $L^\infty \& BMO$:Critical; C^α: Sub-critical.

- Indeed, the regularity theory of (1) in sub-critical setting can be analyzed by unified approaches.
- In sub-critical case, it is almost equivalent to $a(t,x)$ is Hölder continuous.

- Indeed, the regularity theory of (1) in sub-critical setting can be analyzed by unified approaches.
- In sub-critical case, it is almost equivalent to $a(t,x)$ is Hölder continuous.
- By Levi's parametrix method, (1) admits a fundamental solution $\Gamma(\cdot;\cdot)$ such that

$$u(t,x) = \int_{\mathbb{R}} \Gamma(t,x;y,0)u_0(y)dy + \int_0^t \int_{\mathbb{R}} \Gamma(t,x;y,s)f(s,y)dyds.$$

  The kernel satisfies Gaussian bounds (A. Friedman, 1983).
- Hence, the regularity of the solution can be analyzed directly from the explicit expression.

Goals

- Goals: Under the critical regularity setting, establishing the optimal regularity theory of (1) (In the sense of heat eq.).

- Applications: Regularity estimates for semigroups with rough coefficients, with applications to nonlinear and quasilinear equations.

$W^{2,p}$ Estimates in Critical Setting ($u_0 \equiv 0$)

$W^{2,p}$ Esti: For $u_0 \equiv 0$, whether

$$\|u_t\|_{L^p_{t,x}} + \|\partial_x^2 u\|_{L^p_{t,x}} \lesssim \|u\|_{L^p_{t,x}} + \|f\|_{L^p_{t,x}}, \quad \text{(ND)},$$

$$\|u_t\|_{L^p W_x^{-1,p}} + \|\partial_x u\|_{L^p_{t,x}} \lesssim \|u\|_{L^p_{t,x}} + \|f\|_{L^p_t W_x^{-1,p}}, \quad \text{(D)}.$$

- $a(t,x) \in C^0$: Ladyzhenskaya–Solonnikov–Uralceva (1968).
- $a(t,x) \in L^\infty_{t,x} \cap VMO_{t,x}$: Bramanti–Cerutti (1993).
- $a(t,x) \in L^\infty_{t,x} \cap VMO_x$: Krylov (2007).
- Partially VMO: Kim–Krylov (2007), Dong (2010).

Our: optimal Sobolev estimates for the critical Cauchy problem.

Assumptions on the Diffusion Coefficient

Fix $T > 0$, $0 < s < \frac{1}{2}$ and $p = 1/s$. Write

$$\|b\|_{X_s} := \|\partial_x S_0 b\|_{L^{p,\infty}} + \big\| |\partial_x|^s (I - S_0) b \big\|_{L^{p,\infty}}.$$

(H1) a is measurable and $a(t,x) \geq c_0 > 0$ on $[0,T] \times \mathbb{R}$.

(H2) $a(t,\cdot) \in \mathcal{S}'(\mathbb{R})$ and, for every $\phi \in \mathcal{S}(\mathbb{R})$,

$$\sup_{t \in [0,T]} |\langle a(t,\cdot), \phi \rangle| \leq C_\phi.$$

$(\mathrm{H3})_{\delta,s,D}$ The full critical seminorm is bounded:

$$\sup_{t \in [0,T]} \|a(t)\|_{X_s} \leq D.$$

There is a frequency index $N \geq 0$ such that

$$\sup_{t \in [0,T]} \big\| |\partial_x|^s (I - S_N) a(t) \big\|_{L^{p,\infty}} \leq \delta.$$

Only the high-frequency tail is small; the full coefficient need not be.

Discontinuous and Unbounded Coefficients

The assumptions do not require continuity or a global upper bound.

- A jump coefficient:

$$a_J(x) = c_0 + \lambda \mathbf{1}_{\{x>0\}}, \qquad \left||\partial_x|^s a_J(x)\right| \lesssim_s \lambda |x|^{-s}.$$

  The critical tail is controlled by the jump size. Choose $\lambda > 0$ small enough for (H3); moving the cutoff alone does not make an arbitrary jump small.

- An unbounded coefficient:

$$a_U(x) = c_0 + \lambda \log(1 + x^2), \qquad \lambda > 0.$$

  It has the required critical regularity; suitable parameters satisfy (H3). The positive constant is needed for uniform ellipticity.

The approximation must handle both jumps and growth at spatial infinity.

Main Result: Non-Divergence Form

Let $0 < s < \frac{1}{2}$, let u_0 be continuous, and assume

$$u_0 \in \dot{H}^1(\mathbb{R}) \cap \dot{H}^{1+s}(\mathbb{R}), \qquad f \in L^2(0,T;H^s(\mathbb{R})).$$

Theorem (Y., 2026)

There is $\delta_0 = \delta_0(c_0, s, D) > 0$ such that, under (H1), (H2), and $(\mathrm{H3})_{\delta_0,s,D}$, equation (1) has a unique strong solution on $[0,T]$, with

$$\sup_{0\le t\le T} \|u_x(t)\|_{H^s}^2 + \int_0^T \|u_{xx}(t)\|_{H^s}^2\,dt \lesssim \|u_0\|_{\dot{H}^1\cap\dot{H}^{1+s}}^2 + \|f\|_{L^2(0,T;H^s)}^2.$$

Write $\mathbb{X}_s = \{u : u_x \in L^\infty(0,T;H^s),\ u_{xx} \in L^2(0,T;H^s)\}$. The derivative estimate matches the heat equation; the range $s < 1/2$ is sharp.

Sharpness for the regularity index $s < \frac{1}{2}$

Indeed, the above regularity estimate for the solution is optimal, even for the simple case of jumping coefficient and smooth data.

$$\begin{cases} \partial_t v - a(x)\partial_x^2 v = 0, \\ v|_{t=0} = v_0(x) \in C_c^\infty(\mathbb{R}), \end{cases} \qquad a(x) = \begin{cases} 1, & x < 0, \\ 2, & x > 0. \end{cases} \tag{3}$$

Assume $v(t,x) \in \mathbb{X}_s$ for some $s \geq \frac{1}{2}$; By the uniqueness in $\left(L_t^\infty \dot{H}_x^1\right) \cap \left(L_t^2 \dot{H}_x^2\right)$, $v(t,x)$ can be expressed explicitly.

- For $x > 0$:

$$v(t,x) = \int_{-\infty}^{0} \left[(4\sqrt{2}-4) \frac{1}{\sqrt{8\pi t}} \exp\left(-\frac{(x-\sqrt{2}y)^2}{8t} \right) \right] v_0(y)dy$$
$$+ \int_{0}^{\infty} \left[\frac{1}{\sqrt{8\pi t}} \exp\left(-\frac{(x-y)^2}{8t} \right) + (2\sqrt{2}-3) \frac{1}{\sqrt{8\pi t}} \exp\left(-\frac{(x+y)^2}{8t} \right) \right] v_0(y)dy. \tag{2.1}$$

- For $x < 0$:

$$v(t,x) = \int_{0}^{\infty} \left[(2-\sqrt{2}) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(\sqrt{2}x-y)^2}{8t} \right) \right] v_0(y)dy$$
$$+ \int_{-\infty}^{0} \left[\frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t} \right) + (3-2\sqrt{2}) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x+y)^2}{4t} \right) \right] v_0(y)dy \tag{2.2}$$

- However, in general cannot belong to $\dot{H}_x^{2+s}$ for any $t > 0$ and $s \geq \frac{1}{2}$, even for $v_0(x) \in C_c^\infty(\mathbb{R})$.

Step 1: Smooth and Truncate the Coefficient

For $0 < \epsilon \le \min\{1/10, (2c_0)^{-1}\}$, set

$$\bar{a}_\epsilon = \widetilde{\eta}_\epsilon *_{t,x} a, \qquad a^\epsilon = \chi_\epsilon(\bar{a}_\epsilon).$$

Choose χ_ϵ smooth, monotone and odd, with $\chi_\epsilon(z) = z$ for $|z| \le \epsilon^{-1}$, $\chi_\epsilon(z) = 2\epsilon^{-1}$ for $z \ge 2\epsilon^{-1}$, with

$$\|\chi_\epsilon'\|_\infty \lesssim 1, \qquad \|\chi_\epsilon''\|_\infty \lesssim \epsilon, \qquad c_0 \le a^\epsilon \le 2\epsilon^{-1}.$$

$$u_0^\epsilon = \kappa_\epsilon(\eta_\epsilon *_x u_0), \qquad f^\epsilon = \kappa_\epsilon(\widetilde{\eta}_\epsilon *_{t,x} f), \qquad \kappa_\epsilon(x) = \kappa(\epsilon x).$$

Solve the classical approximation problem

$$\partial_t u^\epsilon - \mathcal{L}^\epsilon u^\epsilon = f^\epsilon, \quad u^\epsilon(0) = u_0^\epsilon, \quad \mathcal{L}^\epsilon u = a^\epsilon u_{xx} \text{ or } \partial_x(a^\epsilon u_x).$$

Mollifier gives smoothness; truncation handles unbounded coefficient.

What Must Survive the Approximation?

Put $p = 1/s$, $0 < s < 1/2$, and

$$\|b\|_{X_s} := \|\partial_x S_0 b\|_{L^{p,\infty}} + \left\||\partial_x|^s (I - S_0) b\right\|_{L^{p,\infty}}, \qquad \sup_t \|a(t)\|_{X_s} \le D.$$

Assume $\sup_t \left\||\partial_x|^s (I - S_N) a\right\|_{L^{p,\infty}} \le \delta_0$. The construction preserves, uniformly in ϵ and t,

$$\left\||\partial_x|^s (I - S_0) a^\epsilon\right\|_{L^{p,\infty}} \lesssim_s D,$$
$$\|\partial_x S_0 a^\epsilon\|_{L^{p,\infty}} \lesssim_s D + D^2.$$

Moreover, for a single $K = K(s, N, D, \delta_0) > N + 20$,

$$\boxed{\sup_{\epsilon,t} \left\||\partial_x|^s (I - S_K) a^\epsilon\right\|_{L^{p,\infty}} \le C_s \delta_0.}$$

Nonlinear truncation creates frequencies: convolution estimates alone do not suffice.

Tail Preservation I: Dyadic Composition Estimate

Write $b = \bar{a}_\epsilon$ and $\mathcal{M}^{(2)} = \mathcal{M} \circ \mathcal{M}$. For $j > N + 10$, split

$$\Delta_j \chi_\epsilon(b) = \Delta_j \chi_\epsilon(S_j b) + \Delta_j \big[\chi_\epsilon(b) - \chi_\epsilon(S_j b)\big].$$

The first term gains $2^{-j}\partial_x$; the second uses $\|\chi_\epsilon'\|_\infty \lesssim 1$. Thus

$$|\Delta_j a^\epsilon| \lesssim \underbrace{2^{-j}\mathcal{M}(\partial_x S_2 b)}_{A_{j,1}} + \underbrace{\sum_{2 \le k < N+2} 2^{k-j}\mathcal{M}^{(2)}(\Delta_k b)}_{A_{j,2}}$$
$$+ \underbrace{\sum_{k \ge N+2} \min\{2^{k-j}, 1\}\mathcal{M}^{(2)}(\Delta_k b)}_{A_{j,3}}.$$

Three contributions: very low frequencies, middle frequencies, and the small tail.

Tail Preservation II: Summing the Three Terms

Littlewood–Paley theory and the Lorentz Fefferman–Stein inequality give

$$\big\| |\partial_x|^s (I - S_K) a^\epsilon \big\|_{L^{p,\infty}} \lesssim \sum_{\ell=1}^{3} \left\| \left(\sum_{j \geq K-2} |2^{js} A_{j,\ell}|^2 \right)^{1/2} \right\|_{L^{p,\infty}} =: I_1 + I_2 + I_3.$$

$$I_1 \lesssim_s D\, 2^{-(1-s)K}, \qquad I_2 \lesssim_s D\, 2^{-(1-s)(K-N)}.$$

For the tail, the dyadic kernel is summable:

$$h_{k-j} = \min\{2^{(1-s)(k-j)}, 2^{-s(k-j)}\}, \qquad \sum_{m \in \mathbb{Z}} h_m < \infty.$$

Schur's test and Fefferman–Stein yield

$$I_3 \lesssim_s \big\| |\partial_x|^s (I - S_N) \bar{a}_\epsilon \big\|_{L^{p,\infty}} \lesssim_s \delta_0.$$

Choose K so $I_1 + I_2 \lesssim \delta_0$; this choice is independent of ϵ.

Low-Frequency Regularity: The Uniform Bound

Let $b = \bar{a}_\epsilon$, $c = \chi'_\epsilon(b)$ and $p = 1/s$. The same composition argument controls the high-frequency part of the X_s seminorm. For the low frequencies,

$$\partial_x S_0 a^\epsilon = S_0(c\,\partial_x b).$$

The low–low term is $O(D)$. Only comparable high frequencies can interact to reach S_0, giving

$$\begin{aligned}\|\partial_x S_0 a^\epsilon\|_{L^{p,\infty}} &\lesssim_s D + D\sum_{q\geq 1} 2^{q(1-s)}\|\Delta_q c\|_\infty \\ &\lesssim_s D + D\|\partial_x c\|_\infty.\end{aligned}$$

Split the mollified derivative at $2^k \simeq \epsilon^{-1}$:

$$\|\partial_x \bar{a}_\epsilon\|_\infty \lesssim_s \epsilon^{-1} D, \qquad \|\partial_x c\|_\infty \leq \|\chi''_\epsilon\|_\infty \|\partial_x \bar{a}_\epsilon\|_\infty \lesssim_s D.$$

$$\boxed{\|\partial_x S_0 a^\epsilon\|_{L^{p,\infty}} \lesssim_s D + D^2.}$$

Mollifier Scale: A Uniform Derivative Bound

Choose J_ϵ with $2^{J_\epsilon} \simeq \epsilon^{-1}$ and put $p' = 1/(1-s)$. The space–time mollifier satisfies

$$\|\widetilde{\eta}_\epsilon\|_{L_t^1 L_x^{p',1}} \lesssim \epsilon^{-s}, \qquad \|\partial_x \widetilde{\eta}_\epsilon\|_{L_t^1 L_x^{p',1}} \lesssim \epsilon^{-1-s}.$$

Use a derivative on the dyadic block below J_ϵ, and on the mollifier above J_ϵ. Lorentz Young's inequality gives

$$\begin{aligned}\|\partial_x \bar{a}_\epsilon\|_\infty &\lesssim_s D\left[1 + \epsilon^{-s}\sum_{0\le k\le J_\epsilon} 2^{(1-s)k} + \epsilon^{-1-s}\sum_{k>J_\epsilon} 2^{-sk}\right] \\ &\lesssim_s \epsilon^{-1} D.\end{aligned}$$

Thus the derivative lost in mollification is exactly compensated by the second derivative of the value cutoff:

$$\|\partial_x(\chi_\epsilon'(\bar{a}_\epsilon))\|_\infty \le \|\chi_\epsilon''\|_\infty \|\partial_x \bar{a}_\epsilon\|_\infty \lesssim_s \underbrace{\epsilon \cdot \epsilon^{-1}}_{1} D.$$

This cancellation is the source of the uniform $D + D^2$ bound.

Step 2: Energy Identities (Non-Divergence Form)

Testing the approximate equation by $-u^\epsilon_{xx}$ gives

$$\sup_t \|u^\epsilon_x(t)\|_2^2 + \int_0^T \|u^\epsilon_{xx}\|_2^2 \, dt \lesssim_{c_0} \|u_0'\|_2^2 + \|f\|^2_{L^2_{t,x}} + o_\epsilon(1).$$

This estimate will control the finite low-frequency range. For $j \geq J := K + L$, write

$$\partial_t \Delta_j u^\epsilon - a^\epsilon \partial_{xx} \Delta_j u^\epsilon = \Delta_j f^\epsilon + [\Delta_j, a^\epsilon] u^\epsilon_{xx}.$$

Testing by $-2^{2js}\partial_{xx}\Delta_j u^\epsilon$ yields

$$\frac{1}{2}\frac{d}{dt}\left(2^{2js}\|\partial_x \Delta_j u^\epsilon\|_2^2\right) + 2^{2js}\int a^\epsilon |\partial_{xx}\Delta_j u^\epsilon|^2 \, dx$$
$$= -2^{2js}\int \partial_{xx}\Delta_j u^\epsilon \left(\Delta_j f^\epsilon + [\Delta_j, a^\epsilon] u^\epsilon_{xx}\right) dx.$$

Commutator Formulation

Commutator terms come in:

$$
\begin{aligned}
&[\Delta_j, a^\epsilon] u^\epsilon_{xx} \\
&\quad = \sum_{|l|\le 5} \Delta_j \left(S_{j+l-1} a^\epsilon \Delta_{j+l} \partial_x^2 u^\epsilon \right) - a^\epsilon(x) \partial_x^2 \Delta_j u^\epsilon \\
&\qquad + \sum_{|l|\le 5} \Delta_j \left(S_{j+l-1} \partial_x^2 u^\epsilon \Delta_{j+l} a^\epsilon \right) + \sum_{k \ge j-5, |q-k|\le 1} \Delta_j \left(\Delta_k a^\epsilon \Delta_q \partial_x^2 u^\epsilon \right) \\
&\quad = (S_{j-6} - \mathrm{Id}) a^\epsilon \Delta_j \partial_x^2 u^\epsilon + \sum_{|l|\le 5} [\Delta_j, S_{j-6} a^\epsilon] \Delta_{j+l} \partial_x^2 u^\epsilon \\
&\qquad + \sum_{|l|\le 5} \Delta_j \left(S_{j+l-1} \partial_x^2 u^\epsilon \Delta_{j+l} a^\epsilon \right) + \sum_{k \ge j-5, |q-k|\le 1} \Delta_j \left(\Delta_k a^\epsilon \Delta_q \partial_x^2 u^\epsilon \right) \\
&\qquad + \sum_{-5 < l \le 5} \sum_{q=-6}^{l-2} \Delta_j \left(\Delta_{j+q} a^\epsilon \Delta_{j+l} \partial_x^2 u^\epsilon \right)
\end{aligned}
$$

Rmk: The orange term determines the best Sobolev estimates we can achieve!

The Critical Product Estimate

Set $p = 1/s$, $q = 2/(1-2s)$, so $1/p + 1/q + 1/2 = 1$. The relevant Lorentz-space estimates are

$$H^s(\mathbb{R}) \hookrightarrow L^{q,2}(\mathbb{R}), \qquad \|ABC\|_{L^1} \lesssim \|A\|_{L^{p,\infty}} \|B\|_{L^{q,2}} \|C\|_2.$$

Put $v = u^\epsilon_{xx}$ and $J = K + L$. A representative term is

$$\sum_{j \geq J} 2^{2js} \int \Delta_j v \, \Delta_j \big(S_{j-1} v \, \Delta_j a^\epsilon \big) \, dx.$$

Use dyadic Cauchy–Schwarz, Lorentz Hölder, and Fefferman–Stein to bound the absolute value by

$$\left\| \left(\sum_{j \geq J} 2^{2js} |\Delta_j a^\epsilon|^2 \right)^{1/2} \right\|_{L^{p,\infty}} \|v\|_{L^{q,2}} \|v\|_{H^s} \lesssim_s \delta_0 \|v\|_{H^s}^2.$$

Why Both Preservation Lemmas Are Needed

We need to estimate the dyadic commutator,

$$\|[\Delta_j, b]v\|_2 \lesssim 2^{-j}\|\partial_x b\|_\infty \|v\|_2, \; b = S_{j-6}a^\epsilon.$$

The uniform coefficient bounds imply

$$\|\partial_x S_{j-6}a^\epsilon\|_\infty \lesssim_s (D + D^2)2^K + C_s\delta_0\, 2^j.$$

Hence, for $j \geq K + L$,

$$2^{-j}\|\partial_x S_{j-6}a^\epsilon\|_\infty \lesssim_s (D + D^2)2^{-L} + C_s\delta_0.$$

- The regularity bound controls the large, low-frequency part.
- The tail bound makes the high-frequency contribution small.
- Choose L large and δ_0 small relative to ellipticity; absorb the commutators into the dissipation.

All frequency cutoffs are uniform as $\epsilon \to 0$.

Closing the Uniform Energy Estimate

Set $J = K + L$, $E_J = \sum_{j \geq J} 2^{2js} \|\Delta_j u^\epsilon_x\|_2^2$, and $N_J = \sum_{j \geq J} 2^{2js} \|\Delta_j u^\epsilon_{xx}\|_2^2$. Combining the five paraproduct contributions gives

$$\frac{1}{2} E_J'(t) + c_0 N_J(t) \lesssim_s N_J^{1/2} \|f^\epsilon\|_{H^s} + C_s \left[(D + D^2) 2^{-L} + \delta_0 \right] \|u^\epsilon_{xx}\|_{H^s}^2.$$

Choose δ_0 small and $L \geq 10$ large so that

$$C_s \left[(D + D^2) 2^{-L} + \delta_0 \right] \ll c_0.$$

Absorb the high frequencies; control frequencies below J by the basic energy estimate. Thus, uniformly in ϵ,

$$\sup_t \|u^\epsilon_x(t)\|_{H^s}^2 + \int_0^T \|u^\epsilon_{xx}\|_{H^s}^2 \, dt \lesssim \|u_0\|_{\dot{H}^1 \cap \dot{H}^{1+s}}^2 + \|f\|_{L^2_t H^s_x}^2 + o_\epsilon(1).$$

Step 3: Local Compactness and the Limit Equation

Since a may be unbounded, work on $I_R = [-R, R]$. The coefficient and energy bounds give

$$a^\epsilon \to a \text{ in } L^2((0,T) \times I_R), \qquad \|\partial_t u^\epsilon\|_{L^2(0,T;L^2(I_R))} \leq C_R.$$

Here $a \in L_t^\infty L_{\text{loc}}^{1/s,\infty}$ and $1/s > 2$. Along a subsequence, compactness yields

$$\begin{aligned} u^\epsilon &\to u && \text{uniformly on } [0,T] \times I_R, \\ u_x^\epsilon &\rightharpoonup^* u_x && \text{in } L^\infty(0,T;H^s), \\ u_{xx}^\epsilon &\rightharpoonup u_{xx} && \text{in } L^2(0,T;H^s). \end{aligned}$$

For compactly supported test functions, split

$$a^\epsilon u_{xx}^\epsilon - a u_{xx} = (a^\epsilon - a) u_{xx}^\epsilon + a(u_{xx}^\epsilon - u_{xx}).$$

Strong local convergence of a^ϵ and weak convergence of u_{xx}^ϵ identify the product. Local uniform convergence preserves $u(0,x) = u_0(x)$.

Step 4: Uniqueness by Spatial Localization

Let w be the difference of two solutions, with $w(0) = 0$. Take $\zeta_R = 1$ on I_R, supported in I_{2R}, with $|\partial_x^k \zeta_R| \lesssim R^{-k}$. Test $w_t - a w_{xx} = 0$ by $-\partial_x(\zeta_R^2 w_x)$. Writing $a_L = S_0 a$, $a_H = (I - S_0)a$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int w_x^2 \zeta_R^2 \, dx + \int a\, w_{xx}^2 \zeta_R^2 \, dx \\
&= -2\int a_H w_x w_{xx} \zeta_R \zeta_R' \, dx \\
&\quad + \int w_x^2 \left[a_L' \zeta_R \zeta_R' + a_L (\zeta_R')^2 + a_L \zeta_R \zeta_R'' \right] dx \\
&\lesssim R^{-1} \left(\|w_x\|_{H^s} \|w_{xx}\|_2 + \|w_x\|_2^2 \right).
\end{aligned}$$

The estimate uses $\|a_H\|_{L^{1/s,\infty}} \lesssim D$, $\|a_L'\|_\infty \lesssim D$, and $|a_L(t,x)| \le C + D|x|^{1-s}$.

$$\sup_{t \le T} \int w_x^2 \zeta_R^2 \, dx \lesssim R^{-1} \longrightarrow 0.$$

Thus $w_x = 0$; the equation and $w(0) = 0$ give $w = 0$.

Main Result: Divergence Form

$$\begin{cases} u_t - \partial_x(a(t,x)u_x) = f, \\ u(0,x) = u_0(x), \end{cases} \qquad u_0 \in H^s(\mathbb{R}), \quad f \in L^2(0,T;H^{s-1}(\mathbb{R})). \quad (5)$$

Theorem (Y., 2026)

For $0 < s < \frac{1}{2}$, there is $\delta_0 = \delta_0(c_0, s, D) > 0$ such that (H1), (H2), and $(\mathrm{H3})_{\delta_0,s,D}$ yield a unique weak solution on $[0,T]$ satisfying

$$\sup_{0\le t\le T} \|u(t)\|_{H^s}^2 + \int_0^T \|u_x(t)\|_{H^s}^2 \, dt \lesssim \|u_0\|_{H^s}^2 + \|f\|_{L^2(0,T;H^{s-1})}^2.$$

The same coefficient approximation and preservation lemmas apply. The basic L^2 energy estimate supplies the low-frequency control.

Thank you for your attention!

