数学物理方程讲义

CSSE60003.01

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Abstract

This document collects notes for the course CSSE60003.01 (数学物理方程). Content is organized by topic into six parts: classical elliptic equations (with finite-difference methods), heat and wave equations (with finite-difference methods), Sobolev spaces, weak solutions and regularity (including Galerkin finite elements), Fourier spectral methods, and phase-field models and fluids.

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Part I
Classical elliptic equations

1 Introduction to PDEs

1.1 Course outline

- **Part I.** Classical elliptic equations
- **Part II.** Heat and wave equations
- **Part III.** Sobolev spaces
- **Part IV.** Weak solutions and regularity (Galerkin finite elements)
- **Part V.** Fourier spectral methods
- **Part VI.** Phase-field models and fluids

1.2 Examples of linear PDEs

0. Transport equation: $u_t + b \cdot \nabla u = 0$
1. Laplace equation: $\Delta u = 0$
2. Heat equation: $u_t = \Delta u$
3. Wave equation: $u_{tt} = \Delta u$

Remark 1.1 (Linear operator / equation)**.** An operator L is linear if

$$L(u+v) = L(u) + L(v), \qquad L(cu) = c\,L(u).$$

A linear equation has the form $L(u) = f(x)$.

1.3 Examples of nonlinear PDEs

$$
\begin{aligned}
|\nabla u| &= 1 && \text{(Eikonal)}\\
\Delta u + u^2 &= 0 && \text{(Emden–Fowler)}\\
\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) &= 0 && \text{(p-Laplacian)}\\
u_t + u\,u_x &= 0 && \text{(Burgers)}\\
u_{tt} - \Delta u + \sin u &= 0 && \text{(Klein–Gordon)}\\
iu_t - \Delta u - u^3 &= 0 && \text{(NLS)}\\
\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) &= 0 && \text{(minimal surface)}
\end{aligned}
$$

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \qquad \text{(Navier–Stokes)}$$

1.4 Guiding questions

- Existence & uniqueness
- What do we mean by a solution?
- Stability of IVP & BVP
- Other properties

Definition 1.2 (Classical / weak solutions)**. Classical solutions:** all derivatives appearing in the equation are continuous and the equation holds pointwise.

Weak solutions: the equation holds in a weak sense (prerequisites: L^p spaces).

1.5 Transport equation

Consider

$$\begin{cases} \partial_t u + \vec{b} \cdot \nabla u = f(t,x), \\ u(0,x) = g(x), \end{cases} \qquad u : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}. \tag{1}$$

1.5.1 Method of characteristics

Along characteristic curves,

$$\frac{dt}{1} = \frac{dx_j}{b_j}, \qquad \dot{x}(t) = \vec{b}, \qquad \frac{d}{dt}u\big(x(t),t\big) = \dot{x}(t) \cdot \nabla u + \partial_t u.$$

Parametrize initial data by $(x_0(\xi), t_0(\xi))$. With parameter s,

$$\begin{cases} \dfrac{dt}{ds} = 1, & t(0) = 0, \\ \dfrac{dx_j}{ds} = b_j, & x_j(0) = \xi_j, \\ \dfrac{du}{ds} = f(x,t), & u(0) = g(\xi). \end{cases} \tag{2}$$

Hence $t = s$, $x_j = b_j s + \xi_j$ so $\xi = x - bt$, and

$$u(t,x) = g(x - \vec{b}\,t) + \int_0^t f\big(s, \vec{b}(s-t) + x\big)\,ds. \tag{3}$$

Existence and uniqueness hold provided $b \in C^1$.

Derivation via $u(t, x(t))$. Set $\dot{x}(t) = \vec{b}$, so $x(t) = \vec{b}\,t + \vec{c}$ with $x(0) = \vec{c}$. Then

$$\frac{d}{dt}u = f, \qquad u\big(t, x(t)\big) = u(0) + \int_0^t f\big(s, x(s)\big)\,ds.$$

With $x(s) = \vec{b}\,s + \vec{c} = \vec{b}\,s + x - \vec{b}\,t$ and $u(0) = g(\vec{c}) = g(x - \vec{b}\,t)$, one recovers (3).

1.6 Wave equation via the method of characteristics

Consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & c > 0, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x). \end{cases} \tag{4}$$

Factorization:

$$u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)(\partial_t u - c\partial_x u) = 0.$$

Set $v := \partial_t u - c\partial_x u$. Then

$$\begin{cases} \partial_t v + c\partial_x v = 0, \\ v(0,x) = u_1(x) - cu_0'(x). \end{cases} \tag{5}$$

By the homogeneous transport formula ($f = 0$),

$$v(t,x) = u_1(x - ct) - cu_0'(x - ct).$$

Hence

$$\partial_t u - c\partial_x u = v = u_1(x - ct) - cu_0'(x - ct).$$

Apply the transport formula again with forcing $f(t,x) = u_1(x - ct) - cu_0'(x - ct)$ and initial data u_0:

$$u = u_0(x + ct) + \int_0^t f\big(s, x - c(s - t)\big)\, ds.$$

Change variables $y = x + ct - 2cs$ (so $ds = -\frac{1}{2c}\, dy$), yielding d'Alembert's formula

$$u(t,x) = \frac{1}{2}\big(u_0(x + ct) + u_0(x - ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\, dy. \tag{6}$$

1.7 Laplace and Poisson equations

$$-\Delta u = 0 \quad \text{(Laplace)}, \qquad -\Delta u = f \quad \text{(Poisson)}.$$

Classical solution: $u \in C^2$, $f \in C^0$, equation holds pointwise.

Aim. Solution formula in a bounded domain Ω with suitable boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Existence & uniqueness in the class of continuous functions (to be discussed).

1.7.1 Laplace equation in $\mathbb{R}^n$; radial solutions

In spherical coordinates,

$$\Delta u = u_{rr} + \frac{n-1}{r}u_r + \frac{1}{r^2}\Delta_{S^{n-1}}u,$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on S^{n-1} (for $n = 2$, $\Delta_{S^1} = \partial_{\theta\theta}$).

Radial solutions satisfy $u_{rr} + \frac{n-1}{r}u_r = 0$, hence

$$u = \begin{cases} C_1 r^{2-n} + C_2, & n > 2, \\ C_1 \log r + C_2, & n = 2. \end{cases} \tag{8}$$

1.7.2 Newtonian potential

For f suitably decaying / compactly supported,

$$V(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} f(y)\, |x - y|^{2-n}\, dy = \big(f * |\cdot|^{2-n}\big)(x), & n > 2, \\ \displaystyle\int_{\mathbb{R}^n} f(y)\, \log|x - y|\, dy, & n = 2. \end{cases} \tag{9}$$

Theorem 1.3. *Let $f \in C_c^1(\mathbb{R}^n)$. Then $V \in C^2(\mathbb{R}^n)$ and*

$$\begin{aligned} \Delta V &= -(n-2)\,\omega_n\, f(x), && x \in \mathbb{R}^n,\ n \geq 3, \\ \Delta V &= 2\pi\, f(x), && x \in \mathbb{R}^n,\ n = 2, \end{aligned}$$

where $\omega_n = |S^{n-1}|$ and $|B_1| = \omega_n/n$.

Sketch for $n=3$. Differentiate under the integral (justified by cutting out a small ball):

$$
\begin{aligned}
V_{x_i} &= \int_{\mathbb{R}^n} f(y)\,\partial_{x_i}\big(|x-y|^{-1}\big)\,dy \\
&= \int_{\mathbb{R}^n\setminus B_\rho(x)} f(y)\,\partial_{x_i}\big(|x-y|^{-1}\big)\,dy + \int_{B_\rho(x)} f_{y_i}(y)\,|x-y|^{-1}\,dy - \int_{\partial B_\rho} f(y)\,|x-y|^{-1}\,n_i\,d\sigma_y,
\end{aligned}
$$

using $\partial_{x_i}|x-y|^{-1} = -\partial_{y_i}|x-y|^{-1}$ and integration by parts. The first and third integrals are C^1; the second is C^1 since $f\in C^1$. Hence

$$
\begin{aligned}
\Delta V = &\int_{\mathbb{R}^n\setminus B_\rho(x)} f(y)\,\Delta_x\big(|x-y|^{-1}\big)\,dy \\
&+\int_{B_\rho}\sum_{i=1}^n f_{y_i}(y)\,\partial_{x_i}\big(|x-y|^{-1}\big)\,dy - \int_{\partial B_\rho} f(y)\sum_{i=1}^n \partial_{x_i}\big(|x-y|^{-1}\big)\,n_i\,d\sigma.
\end{aligned}
$$

Away from the diagonal, $\Delta_x|x-y|^{-1}=0$, so $\Delta V = I_1+I_2$ with

$$
\begin{aligned}
|I_1| &\le C\int_{B_\rho(x)}\frac{1}{|x-y|^2}\,dy \le C\rho \to 0 \quad (\rho\to 0),\\
I_2 &= -\int_{\partial B_\rho(x)} f(y)\,\rho^{-2}\,d\sigma = -\omega_n\,f(\bar y),
\end{aligned}
$$

where $f(\bar y)$ is the spherical mean of f on $\partial B_\rho(x)$. Thus $\lim_{\rho\to 0} I_2 = -\omega_n f(x)$.

Other dimensions: exercise (same idea). □

Sketch for $n=2$. With $V(x)=\int_{\mathbb{R}^2} f(y)\log|y-x|\,dy$,

$$
\Delta V = \sum_{i=1}^2\int_{\mathbb{R}^2} f(y)\,\partial_{x_ix_i}\log|y-x|\,dy.
$$

Since $\Delta_x\log|y-x|=0$ for $y\ne x$, cutting out $B_\varepsilon(x)$ yields

$$
\Delta V = \sum_{i=1}^2\int_{B_\varepsilon(x)} f(y)\,\partial_{x_ix_i}\log|y-x|\,dy.
$$

Using $\partial_{x_i}\log|y-x| = -\partial_{y_i}\log|y-x|$ and integration by parts,

$$
\Delta V = \sum_{i=1}^2\left(\int_{B_\varepsilon(x)}\partial_{y_i}f(y)\,\partial_{x_i}\log|y-x|\,dy - \int_{\partial B_\varepsilon(x)} f(y)\,\partial_{x_i}\log|y-x|\,n_i\,d\sigma_y\right) =: I_1+I_2.
$$

Recall $\partial_{x_i}\log|y-x| = -(y_i-x_i)/|y-x|^2$. Then

$$
|I_1| \le C\|f\|_{C^1}\int_{B_\varepsilon(x)}\frac{1}{|y-x|}\,dy = C\|f\|_{C^1}\cdot 2\pi\varepsilon\to 0,
$$

and $(y-x)\cdot\vec n=\varepsilon$ on $\partial B_\varepsilon(x)$ implies

$$
I_2 = \int_{\partial B_\varepsilon(x)} f(y)\,\varepsilon^{-1}\,d\sigma_y.
$$

Thus

$$
\Big|I_2 - 2\pi f(x)\Big| = \Big|\varepsilon^{-1}\int_{\partial B_\varepsilon(x)}\big(f(y)-f(x)\big)\,d\sigma_y\Big| \le C\varepsilon\to 0,
$$

so $\lim_{\varepsilon\to 0} I_2 = 2\pi f(x)$. □

Remark 1.4. 1. The result holds for $f \in C_c^\alpha(\mathbb{R}^n)$.

2. It also holds under decay $|f(y)| \le C(1+|y|)^{-l}$ with $l > 2$.

Exercise 1.5.

$$\int \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|)^l}\, dy \lesssim \begin{cases} (1+|x|)^{2-l}, & 2 < l < n, \\ (1+|x|)^{2-n} \log|x|, & l = n, \\ (1+|x|)^{2-n}, & l > n. \end{cases}$$

Next aim. General solution formula for

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with $f, g \in C^0$; prove existence and uniqueness.

1.7.3 Properties of harmonic functions

Theorem 1.6 (Mean-value property). *If $u \in C^2(\Omega)$ and $\Delta u = 0$, then for every ball $B_\rho(x) \Subset \Omega$,*

$$u(x) = \frac{1}{|\partial B_\rho(x)|} \int_{\partial B_\rho(x)} u\, d\sigma = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} u\, dy. \tag{10}$$

Proof. By the divergence theorem,

$$\begin{aligned} \int_{B_\rho} \Delta u &= \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}\, d\sigma = \rho^{n-1} \int_{|\omega|=1} \partial_\rho u(x+\rho\omega)\, d\sigma \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega)\, d\sigma. \end{aligned}$$

If $\Delta u = 0$, the spherical mean is independent of ρ. Integrating from 0 to r,

$$\int_{|\omega|=1} u(x+r\omega)\, d\sigma = u(x)\, \omega_n,$$

hence

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, d\sigma.$$

Integrating in ρ from 0 to r gives the volume form of the MVP:

$$u(x)\, \frac{\omega_n}{n} r^n = \int_{B_r(x)} u(y)\, dy \implies u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy.$$

□

Remark 1.7. If $\Delta u \ge 0$ (i.e. u is subharmonic), then

$$\frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega)\, d\sigma \ge 0,$$

so

$$u(x) \le \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, d\sigma$$

(sub-mean-value inequality). If instead $\Delta u \le 0$, the inequality reverses (superharmonic / super-mean-value). The MVP characterizes harmonic functions.

Theorem 1.8 (MVP $\Rightarrow$ harmonic). *If $u \in C^0(\Omega)$ satisfies the MVP, then u is smooth and harmonic in Ω.*

Proof. Choose a radial mollifier $\varphi \in C_c^\infty(B_1(0))$ with $\int_{B_1} \varphi = 1$, and set $\varphi_\varepsilon(z) = \varepsilon^{-n}\varphi(z/\varepsilon)$. Then for $x \in \Omega_\varepsilon := \{y \in \Omega : d(y, \partial\Omega) > \varepsilon\}$,

$$\begin{aligned}(\varphi_\varepsilon * u)(x) = \int u(x+y)\varphi_\varepsilon(y)\,dy &= \int_{|y|<1} u(x+\varepsilon y)\varphi(y)\,dy \\ &= \int_0^1 \varphi(r)\,r^{n-1}\,dr \int_{|\omega|=1} u(x+\varepsilon r\omega)\,d\sigma = u(x),\end{aligned}$$

by the MVP (and normalization of φ). Thus $u = \varphi_\varepsilon * u$ on Ω_ε, so $u \in C^\infty(\Omega)$ and $\partial^\alpha u = \partial^\alpha \varphi_\varepsilon * u$. Moreover, for every $B_r(x) \Subset \Omega$,

$$\int_{B_r(x)} \Delta u = r^{n-1}\frac{\partial}{\partial r}\big(\omega_n u(x)\big) = 0,$$

hence $\Delta u = 0$ in Ω. □

1.7.4 Consequences of the MVP

Theorem 1.9 (Maximum principle). *If $u \in C^0(\overline{\Omega})$ satisfies the MVP, then u attains its maximum and minimum only on $\partial\Omega$, unless u is constant.*

Proof. Let $M = \max_{\overline{\Omega}} u$ and $\Sigma := \{x \in \Omega : u(x) = M\}$. Continuity implies Σ is relatively closed in Ω. If $x_0 \in \Sigma$ and $\overline{B}_r(x_0) \subset \Omega$, the MVP gives

$$M = u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u \le M,$$

so $u \equiv M$ on $B_r(x_0)$, i.e. Σ is open. Thus $\Sigma = \emptyset$ or $\Sigma = \Omega$: either the maximum is attained only on $\partial\Omega$, or $u \equiv M$. □

Corollary 1.10 (Uniqueness for the Dirichlet problem). *Solutions of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

are unique in $C^0(\overline{\Omega}) \cap C^2(\Omega)$.

Theorem 1.11 (Gradient estimate). *Let $u \in C(\overline{B}_R)$ be harmonic in $B_R(x_0)$. Then*

$$|Du(x_0)| \le \frac{n}{R}\max_{\overline{B}_R}|u|.$$

Proof. Harmonicity implies $u \in C^\infty(B_R)$ and $\Delta(D_{x_i}u) = 0$. By the volume MVP and the divergence theorem,

$$D_{x_i}u(x_0) = \frac{n}{\omega_n R^n}\int_{B_R(x_0)} D_{x_i}u = \frac{n}{\omega_n R^n}\int_{\partial B_R} u\,\nu_i\,d\sigma,$$

which yields the stated bound. □

2 Gradient estimates, Harnack inequality, and Greenś functions

2.1 Gradient estimates

The gradient estimate for harmonic functions yields

$$|D_{x_i}u(x_0)| \le \frac{n}{\omega_n R^n} \max|u| \cdot \omega_n R^{n-1} \le \frac{n}{R} \max|u|.$$

Lemma 2.1. *If $u \ge 0$ and $\Delta u = 0$, then*

$$|Du(x_0)| \le \frac{n}{R} u(x_0).$$

Corollary 2.2. *If $\Delta u = 0$ in $\mathbb{R}^n$ and $u \le M$ for some constant M, then u is constant.*

Corollary 2.3. *If $\Delta u = 0$ in $\mathbb{R}^n$ and $|u| \le (1+|y|)^\ell$ with $\ell < 1$, then $u \equiv C$.*

Theorem 2.4 (Higher-order derivative estimates)**.** *Let $u \in C(\overline{B}_R)$ with $\Delta u = 0$ in B_R. Then*

$$|D^m u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m} \max_{B_R} |u|.$$

Equivalently, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = m$,

$$|D^\alpha u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m} \max_{B_R} |u|.$$

Proof of Lemma 2.1 / Corollary 1. Since $\Delta \partial_{x_i} u = 0$, the volume MVP and the divergence theorem give

$$\begin{aligned}\partial_{x_i} u(x_0) &= \frac{n}{\omega_n R^n} \int_{B_R(x_0)} \partial_{x_i} u \, dy = \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u \, n_i \, d\sigma_y \\ &\le \frac{n}{R} \cdot \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u \, d\sigma_y \le \frac{n}{R} u(x_0).\end{aligned}$$

The same bound holds for $-\partial_{x_i} u$, hence for each component. For the full gradient, take a unit vector ξ maximizing $\xi \cdot Du(x_0)$:

$$|Du(x_0)| = \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u \, (\xi \cdot n) \, d\sigma_y \le \frac{n}{R} u(x_0),$$

where the last step uses $u \ge 0$ and the surface MVP. ☐

Proof of Corollary 2.2. Apply Lemma 2.1 to the nonnegative harmonic function $v := M - u$:

$$|Dv(x_0)| \le \frac{n}{R} v(x_0) \le \frac{n}{R} \big(M - \inf_{\mathbb{R}^n} u\big) \qquad \forall R > 0.$$

(If u is not bounded from below one first replaces M by a local bound on $B_R(x_0)$ and lets $R \to \infty$ after estimating on each ball.) Letting $R \to \infty$ yields $Dv \equiv 0$, hence u is constant. ☐

Proof of Corollary 2.3. For $\ell < 1$,

$$\begin{aligned}|D_k u(x_0)| &\le \frac{n}{R} \max_{x \in B_R(x_0)} |u| \le \frac{n}{R} \max_{x \in B_R(x_0)} (1+|x|)^\ell \\ &\le \frac{C_n}{R} (1+R)^\ell \le C_n R^{\ell-1}.\end{aligned}$$

As $R \to \infty$ we obtain $D_k u(x_0) = 0$, so $u \equiv C$. ☐

Proof of Theorem 2.4. We proceed by induction on m. The case $m = 1$ is the gradient estimate.

Assume the claim holds for m, and consider $m+1$. Fix $\theta \in (0,1)$ and set $r = (1-\theta)R$. Then $B_{R-r}(y) \subset B_R(x_0)$ for every $y \in \overline{B}_r(x_0)$, and

$$|D^{m+1}u(x_0)| \le \frac{n}{r} \max_{\overline{B}_r} |D^m u|.$$

By the maximum principle, $\max_{B_r(x_0)} |D^m u| = \max_{\partial B_r(x_0)} |D^m u|$. Applying the induction hypothesis on balls of radius $R - r = \theta R$ gives

$$\max_{B_r} |D^m u| \le \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{B_R} |u|.$$

Hence

$$\begin{aligned} |D^{m+1}u(x_0)| &\le \frac{n}{r} \cdot \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{B_R} |u| \\ &= \frac{n^{m+1} e^{m-1} m!}{R^{m+1}\theta^m(1-\theta)} \max_{B_R} |u|, \end{aligned}$$

where we used $(R-r)^m r = (\theta R)^m \cdot (1-\theta)R = R^{m+1}\theta^m(1-\theta)$.

Choose $\theta = \dfrac{m}{m+1}$. Then

$$\frac{1}{\theta^m(1-\theta)} = \Big(1 + \frac{1}{m}\Big)^m (m+1) < e(m+1),$$

and therefore

$$|D^{m+1}u(x_0)| \le \frac{n^{m+1}(m+1)!\, e^m}{R^{m+1}} \max_{B_R(x_0)} |u|.$$

This completes the induction. □

2.2 Analyticity of harmonic functions

Theorem 2.5. *Harmonic functions are (real) analytic.*

Proof. Write the Taylor expansion

$$u(x+h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!}\big[(h_1\partial_1 + \cdots + h_n\partial_n)^i u\big](x) + R_m(h),$$

with remainder

$$R_m(h) = \frac{1}{m!}\big[(h_1\partial_1 + \cdots + h_n\partial_n)^m u\big](x_1 + \theta h_1, \ldots, x_n + \theta h_n)$$

for some $\theta \in (0,1)$. Using Theorem 2.4,

$$\begin{aligned} |R_m(h)| &\le \frac{1}{m!} |h|^m n^m \cdot \frac{n^m e^{m-1} m!}{R^m} \max_{B_R} |u| \\ &\le \Big(\frac{|h| n^2 e}{R}\Big)^m \max |u|. \end{aligned}$$

Whenever $|h|n^2 e < R/2$, we have $R_m(h) \to 0$ as $m \to +\infty$. Thus the Taylor series converges to u in a neighborhood of x. □

2.3 Harnack inequality

Theorem 2.6 (Harnack inequality)**.** *Let $\Delta u = 0$ in Ω with $u \geq 0$. Then for every compact $K \subset \Omega$ there exists $C = C(n, K, \Omega)$ such that*

$$\frac{1}{C}\, u(y) \leq u(x) \leq C\, u(y) \qquad \forall x, y \in K.$$

Proof. Let $r = \dfrac{1}{4}\operatorname{dist}(K, \partial\Omega)$. For all $x, y \in K$ with $|x - y| \leq r$ one has $B_r(y) \subseteq B_{2r}(x) \subseteq \Omega$. By the volume MVP,

$$\begin{aligned} u(x) &= \frac{n}{\omega_n (2r)^n} \int_{B_{2r}(x)} u(z)\, dz \geq \frac{n}{\omega_n (2r)^n} \int_{B_r(y)} u(z)\, dz \\ &= \frac{1}{2^n} \cdot \frac{n}{\omega_n r^n} \int_{B_r(y)} u(z)\, dz = \frac{1}{2^n}\, u(y). \end{aligned}$$

Similarly $u(y) \geq \dfrac{1}{2^n} u(x)$, so

$$\frac{1}{2^n}\, u(y) \leq u(x) \leq 2^n u(y) \qquad \forall x, y \in K \text{ with } |x - y| \leq r.$$

Since K is compact, finitely many balls of radius $r/2$ cover K. Chaining the local comparison along a cover of length N yields

$$\frac{1}{2^{nN}}\, u(y) \leq u(x) \leq 2^{nN} u(y) \qquad \forall x, y \in K.$$

□

Remark 2.7. In particular, if $B_{4R}(x_0) \subset \Omega$ and $x, y \in B_R(x_0)$, a direct MVP comparison on balls of radii comparable to R gives a constant $C = C(n)$ (e.g. $C = 3^n$ in a standard configuration). For a general compact K one covers K by balls $B_R(x_i)$ with $4R < \operatorname{dist}(K, \partial\Omega)$ and takes $C = c^N$.

Remark 2.8. Harnack's inequality implies

$$\sup_K u \leq C \inf_K u.$$

Theorem 2.9 (Harnack convergence theorem)**.** *Let $\{u_j\}$ be a monotone increasing sequence of harmonic functions in Ω. Suppose that at some point $y \in \Omega$ the sequence $\{u_j(y)\}$ is bounded. Then $\{u_j\}$ converges uniformly on every compact $K \Subset \Omega$.*

2.4 Green's function

2.4.1 Introduction

Recall the fundamental solution

$$\Gamma(a, x) = \begin{cases} \dfrac{1}{2\pi} \log|x - a|, & n = 2, \\ -\dfrac{1}{(n-2)\omega_n}\, |x - a|^{2-n}, & n \geq 3. \end{cases}$$

We have $\Delta_x \Gamma(a, x) = 0$ for $x \neq a$. Moreover,

$$\int_{\partial B_r(a)} \nabla_x \Gamma(a, x) \cdot \vec{n}\, d\sigma = 1 \qquad \forall r > 0,$$

since $\dfrac{\partial \Gamma}{\partial n} = \nabla \Gamma \cdot n$.

Case $n = 2$.

$$\int_{\partial B_r(a)} \nabla_x \Gamma \cdot \vec{n}\, d\sigma = \frac{1}{2\pi} \int_{\partial B_r(a)} \frac{x-a}{|x-a|^2} \cdot \frac{x-a}{|x-a|}\, d\sigma = \frac{1}{2\pi} \int_{\partial B_r(a)} \frac{1}{|x-a|}\, d\sigma = 1.$$

Case $n \geq 3$.

$$\begin{aligned}\int_{\partial B_r(a)} \nabla_x \Gamma \cdot \vec{n}\, d\sigma &= \frac{n-2}{(n-2)\omega_n} \int_{\partial B_r(a)} \frac{1}{|x-a|^{n-1}} \frac{x-a}{|x-a|} \cdot \frac{x-a}{|x-a|}\, d\sigma \\ &= \frac{1}{\omega_n} \int_{\partial B_r(a)} r^{1-n}\, d\sigma = 1.\end{aligned}$$

Lemma 2.10 (Representation formula)**.** *Suppose Ω is bounded and $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then for every $a \in \Omega$,*

$$u(a) = \int_\Omega \Gamma(a,y)\Delta u(y)\, dy - \int_{\partial\Omega} \Big(\Gamma(a,y) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Gamma}{\partial n}(a,y) \Big)\, d\sigma_y.$$

Proof. We treat the case $n \geq 3$. Apply Green's formula on $\Omega \setminus B_\varepsilon(a)$:

$$\begin{aligned}&\int_{\Omega\setminus B_\varepsilon(a)} (\Gamma\Delta u - u\Delta\Gamma)\, dy \\ &\qquad = \int_{\partial\Omega} \Big(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \Big)\, d\sigma_y - \int_{\partial B_\varepsilon(a)} \Big(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \Big)\, d\sigma_y.\end{aligned}$$

Since $\Delta\Gamma \equiv 0$ in $\Omega \setminus B_\varepsilon(a)$ for every $\varepsilon > 0$,

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(a)} \Big(\underbrace{\Gamma \frac{\partial u}{\partial n}}_{I_1} - \underbrace{u \frac{\partial \Gamma}{\partial n}}_{I_2} \Big)\, d\sigma_y \\ &\qquad = \int_\Omega \Gamma\Delta u\, dy - \int_{\partial\Omega} \Big(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \Big)\, d\sigma_y.\end{aligned}$$

Estimate of I_1.

$$\Big| \int_{\partial B_\varepsilon(a)} \Gamma \frac{\partial u}{\partial n}\, d\sigma_y \Big| \leq \frac{1}{(n-2)\omega_n} \max|\nabla u| \int_{\partial B_\varepsilon(a)} \varepsilon^{2-n}\, d\sigma_y = \frac{\max|\nabla u|}{n-2}\, \varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

Estimate of I_2.

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(a)} u \frac{\partial \Gamma}{\partial n}\, d\sigma_y &= \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(a)} \Big(\frac{1}{\omega_n} u(y)\, \frac{y-a}{|y-a|^n} \cdot \frac{y-a}{|y-a|} \Big)\, d\sigma_y \\ &= \lim_{\varepsilon\to 0} \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(a)} u(y)\, d\sigma_y = \lim_{\varepsilon\to 0} \frac{1}{|\partial B_\varepsilon(a)|} \int_{\partial B_\varepsilon(a)} u(y)\, d\sigma_y = u(a),\end{aligned}$$

as long as $u \in C^1(\Omega)$. Combining the limits yields the claimed formula. □

Remark 2.11. (i) If $a \notin \overline{\Omega}$, the right-hand side of the representation formula vanishes. Indeed $\Delta\Gamma(a,\cdot) \equiv 0$ in Ω (no singularity), so

$$0 \equiv \int_\Omega u\Delta\Gamma\, dy = \int_\Omega \Gamma\Delta u\, dy - \int_{\partial\Omega} \Big(\Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \Big)\, d\sigma_y.$$

(ii) For every $a \in \Omega$, $\Gamma(a,\cdot)$ is integrable on Ω (despite the singularity at $y = a$).

(iii) Taking $u \equiv 1$ gives

$$\int_{\partial\Omega} \frac{\partial \Gamma}{\partial n}\, d\sigma = 1.$$

Derivation of the Green's function. Let $\Omega \subseteq \mathbb{R}^n$ be bounded and $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. For $x \in \Omega$,

$$u(x) = \int_\Omega \Gamma(x,y)\Delta u(y)\,dy - \int_{\partial\Omega} \Big(\Gamma(x,y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial \Gamma}{\partial n}(x,y)\Big)\,d\sigma_y. \tag{11}$$

Assume u solves the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Set $G(x,y) = \Gamma(x,y) + \Phi(x,y)$, where $\Phi(x,\cdot) \in C^2(\overline{\Omega})$ satisfies $\Delta_y \Phi(x,y) = 0$ for $y \in \Omega$. Then

$$0 = \int_\Omega \Phi(x,y)\Delta u(y)\,dy - \int_{\partial\Omega} \Big(\Phi(x,y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial \Phi}{\partial n}(x,y)\Big)\,d\sigma_y. \tag{12}$$

Adding (11) and (12) yields

$$u(x) = \int_\Omega G(x,y)\Delta u(y)\,dy - \int_{\partial\Omega} \Big(G(x,y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G}{\partial n}(x,y)\Big)\,d\sigma_y.$$

Choosing the corrector so that

$$\begin{cases} \Delta_y \Phi(x,y) = 0 & y \in \Omega, \\ \Phi(x,y) = -\Gamma(x,y) & y \in \partial\Omega, \end{cases}$$

we obtain $G(x,y) = 0$ for $y \in \partial\Omega$, and therefore

$$u(x) = -\int_\Omega G(x,y) f(y)\,dy + \int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}(x,y)\,d\sigma_y.$$

In other words, the Green's function satisfies

$$\begin{cases} \Delta G = \delta_x & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega. \end{cases}$$

It remains to solve for Φ (equivalently G) on concrete domains Ω.

2.4.2 Properties of the Green's function

Theorem 2.12 (Symmetry of the Green's function)**.** *$G(x_1, x_2) = G(x_2, x_1)$ for all $x_1 \neq x_2$ in Ω. (The diagonal $x_1 = x_2$ is singular.)*

Proof. Pick $x_1, x_2 \in \Omega$ and $\varepsilon > 0$ small enough that $B_\varepsilon(x_1) \cap B_\varepsilon(x_2) = \emptyset$. Write $G_1(y) = G(x_1, y)$ and $G_2(y) = G(x_2, y)$. Green's second identity on $\Omega \setminus \big(B_\varepsilon(x_1) \cup B_\varepsilon(x_2)\big)$ gives

$$\begin{aligned} &\int_{\Omega\setminus(B_\varepsilon(x_1)\cup B_\varepsilon(x_2))} (G_1 \Delta G_2 - G_2 \Delta G_1)\,dy \\ &\quad = \int_{\partial\Omega} \Big(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\Big)\,d\sigma \\ &\qquad - \int_{\partial B_\varepsilon(x_1)} \Big(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\Big)\,d\sigma \\ &\qquad - \int_{\partial B_\varepsilon(x_2)} \Big(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\Big)\,d\sigma. \end{aligned}$$

Note that

$$\begin{cases} \Delta_y G_1 = \Delta_y G_2 = 0 & \text{in } \Omega \setminus \big(B_\varepsilon(x_1) \cup B_\varepsilon(x_2)\big), \\ G_1 = G_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence

$$0 = \int_{\partial B_\varepsilon(x_1)} \Big(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n} \Big)\, d\sigma_y + \int_{\partial B_\varepsilon(x_2)} \Big(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n} \Big)\, d\sigma_y. \tag{$*$}$$

We check the case $n \geq 3$. First,

$$\int_{\partial B_\varepsilon(x_1)} G_1 \frac{\partial G_2}{\partial n}\, d\sigma_y = \int_{\partial B_\varepsilon(x_1)} \big(\Gamma(x_1,y) + \Phi(x_1,y)\big) \frac{\partial G_2}{\partial n}\, d\sigma_y.$$

Since $\Gamma(x_1,y) = -\dfrac{1}{(n-2)\omega_n}|x_1 - y|^{2-n}$,

$$\int_{\partial B_\varepsilon(x_1)} \frac{1}{(n-2)\omega_n}|x_1-y|^{2-n}\Big|\frac{\partial G_2}{\partial n}\Big|\, d\sigma_y \leq C\,\varepsilon^{n-1}\varepsilon^{2-n} \max|DG_2| = C\varepsilon \to 0,$$

$$\int_{\partial B_\varepsilon(x_1)} |\Phi(x_1,y)|\,\Big|\frac{\partial G_2}{\partial n}\Big|\, d\sigma_y \leq C\varepsilon^{n-1} \to 0.$$

Thus

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G_1 \frac{\partial G_2}{\partial n}\, d\sigma_y = 0,$$

and similarly

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_2)} G_2 \frac{\partial G_1}{\partial n}\, d\sigma_y = 0.$$

Next,

$$\int_{\partial B_\varepsilon(x_1)} G_2 \frac{\partial G_1}{\partial n}\, d\sigma_y = \int_{\partial B_\varepsilon(x_1)} G_2(y)\, \nabla_y \big(\Gamma(x_1,y) + \Phi(x_1,y)\big) \cdot \vec{n}\, d\sigma_y.$$

A direct computation gives

$$\nabla_y \Gamma(x_1,y)\cdot\vec{n} = -\frac{(2-n)}{\omega_n(n-2)}\,|y-x_1|^{1-n}\,\frac{y-x_1}{|y-x_1|}\cdot\frac{y-x_1}{|y-x_1|} = \frac{1}{\omega_n}\,|y-x_1|^{1-n}.$$

It is then not hard to see that

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G_2(y)\,\nabla_y\Gamma(x_1,y)\cdot\vec{n}\, d\sigma = G_2(x_1),$$

while

$$\Big|\int_{\partial B_\varepsilon(x_1)} G_2(y)\,\nabla_y\Phi\cdot\vec{n}\, d\sigma_y\Big| \leq \varepsilon^{n-1}\max|G_2|\,\max|\nabla_y\Phi| \to 0.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G_2(y)\frac{\partial G_1}{\partial n}\, d\sigma_y = G_2(x_1).$$

Similarly,

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_2)} G_1(y)\frac{\partial G_2}{\partial n}\, d\sigma_y = G_1(x_2).$$

Passing to the limit in ($*$) yields $G_1(x_2) = G_2(x_1)$, i.e. $G(x_1,x_2) = G(x_2,x_1)$. □

Remark 2.13. (i) $G(x,y) \in C_y^\infty\big(\Omega \setminus \{x\}\big)$.

(ii) $G(x,y) < 0$ for $y \in \Omega\setminus\{x\}$. Indeed, with the convention $\Delta\Gamma = \delta_x$, one has $\Gamma \to -\infty$ as $y \to x$, while $\Phi = -\Gamma$ on $\partial\Omega$ is harmonic in Ω. Away from the pole, G is harmonic, $G = 0$ on $\partial\Omega$, and $G \to -\infty$ at $y = x$, hence the maximum principle yields $G < 0$ in $\Omega\setminus\{x\}$.

Recall the representation

$$u(x) = -\int_\Omega G(x,y)f(y)\,dy + \int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}\,d\sigma_y.$$

Writing $u = u_1 + u_2$, we have the decomposition

$$u_1 \text{ solves } \begin{cases} -\Delta u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad u_2 \text{ solves } \begin{cases} -\Delta u_2 = 0 & \text{in } \Omega, \\ u_2 = g & \text{on } \partial\Omega, \end{cases}$$

and therefore u solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Remark 2.14 (Task)**.** Find $G(x,y)$ / $\Phi(x,y)$ for concrete domains Ω.

2.5 Solving for the Green's function

3 Greenś functions on special domains; Perronś method

3.1 Green's function for the ball

3.1.1 Ball $\Omega = B_R(0)$

Definition 3.1 (Inversion)**.** The inversion of $x \in \mathbb{R}^n \setminus \{0\}$ with respect to $\partial B_R(0)$ is

$$x^* = \frac{R^2}{|x|^2}\,x.$$

This map sends $0 \mapsto \infty$, $\infty \mapsto 0$, fixes $\partial B_R(0)$, and interchanges the interior and exterior of the ball.

Idea. Invert the singularity from $x \in B_R(0)$ to $x^* \notin B_R(0)$. Recall $\Delta_y \Gamma(z,y) = 0$ for all $y \neq z$. Hence, for $n = 3$ (and similarly in general),

$$\Delta_y \Gamma\Big(\frac{|x|}{R}x^*, \frac{|x|}{R}y\Big) = 0 \qquad \text{for } y \neq x^*.$$

Define the image term

$$\Phi(x,y) := -\Gamma\Big(\frac{|x|}{R}x^*, \frac{|x|}{R}y\Big)$$

(so that $G = \Gamma + \Phi$), and obtain the explicit formula

$$G(x,y) = \begin{cases} \dfrac{1}{2\pi}\left(\log|x-y| - \log\Big|\dfrac{R}{|x|}x - \dfrac{|x|}{R}y\Big|\right), & n = 2, \\ -\dfrac{1}{(n-2)\omega_n}\left(|x-y|^{2-n} - \Big|\dfrac{R}{|x|}x - \dfrac{|x|}{R}y\Big|^{2-n}\right), & n \geq 3. \end{cases} \tag{13}$$

Verification. We need to check:

1. $\Delta_y \Phi(x,y) = 0$ for $y \in B_R(0)$. Since $x \in B_R(0)$ implies $x^* \notin \overline{B}_R(0)$, one has $y \neq x^*$ for $y \in B_R(0)$.
2. $\Phi(x,y) = -\Gamma(x,y)$ for $y \in \partial B_R(0)$.

Proof of the boundary condition. Let $y \in \partial B_R(0)$ and $x \neq 0$. Then $|y| = R$ and

$$\begin{aligned}\Big|\frac{|x|}{R}x^* - \frac{|x|}{R}y\Big|^2 &= \frac{|x|^2}{R^2}\,|x^* - y|^2 = \frac{|x|^2}{R^2}\big(|x^*|^2 + |y|^2 - 2x^* \cdot y\big)\\ &= \frac{|x|^2}{R^2}\left(\Big|\frac{R^2}{|x|^2}x\Big|^2 + |y|^2 - 2\frac{R^2}{|x|^2}x \cdot y\right)\\ &= R^2 + |x|^2 - 2x \cdot y = |x - y|^2.\end{aligned}$$

Hence

$$\Gamma\Big(\frac{|x|}{R}x^*, \frac{|x|}{R}y\Big) = \Gamma(x, y) \qquad \text{for } y \in \partial B_R(0),$$

so $\Phi = -\Gamma$ on $\partial B_R(0)$. □

Lemma 3.2 (Normal derivative on $\partial B_R(0)$)**.** *Suppose G is given by (13). Then for all $x \in B_R(0)$ and $y \in \partial B_R(0)$,*

$$\frac{\partial G}{\partial n}(x, y) = \frac{R^2 - |x|^2}{\omega_n R\,|x - y|^n}.$$

Proof. Consider the case $n \geq 3$. From (13),

$$G(x, y) = -\frac{1}{(n-2)\omega_n}\left(|x - y|^{2-n} - \Big|\frac{R}{|x|}x - \frac{|x|}{R}y\Big|^{2-n}\right).$$

Differentiating in y,

$$\nabla_y G = \frac{1}{\omega_n}\left(|x - y|^{1-n}\,\frac{y - x}{|y - x|} - \Big|\frac{R}{|x|}x - \frac{|x|}{R}y\Big|^{1-n}\,\frac{\frac{|x|}{R}y - \frac{R}{|x|}x}{\big|\frac{R}{|x|}x - \frac{|x|}{R}y\big|} \cdot \frac{|x|}{R}\right).$$

On $\partial B_R(0)$ one has $\big|\frac{R}{|x|}x - \frac{|x|}{R}y\big| = |x - y|$, whence

$$\nabla_y G = \frac{1}{\omega_n}\,|x - y|^{-n}\left((y - x) - \Big(\frac{|x|^2}{R^2}y - x\Big)\right) = \frac{1}{\omega_n}\,|x - y|^{-n}\left(y - \frac{|x|^2}{R^2}y\right).$$

Therefore, with outward unit normal $n = y/R$,

$$\frac{\partial G}{\partial n}(x, y) = D_y G \cdot \frac{y}{R} = \frac{1}{\omega_n}\,|x - y|^{-n}\,R\Big(1 - \frac{|x|^2}{R^2}\Big) = \frac{1}{\omega_n}\,|x - y|^{-n}\,\frac{R^2 - |x|^2}{R}.$$

□

Definition 3.3 (Poisson kernel)**.** The function

$$K(x, y) := \frac{\partial G}{\partial n}(x, y)$$

is called the Poisson kernel (for the ball).

Properties.

1. $K(x, y)$ is smooth for $x \neq y$.
2. $K(x, y) > 0$ for all $|x| < R$.
3. $\displaystyle\int_{|y|=R} K(x, y)\,d\sigma_y = 1$ for all $|x| < R$.

Indeed, recalling Green's representation and taking $u \equiv 1$ (so $f = 0$ and $G\big|_{\partial\Omega} = 0$),

$$1 = \int_{\partial\Omega} \frac{\partial G}{\partial n}(x, y)\, d\sigma_y.$$

Theorem 3.4 (Poisson integral formula)**.** *For every $\varphi \in C(\partial B_R(0))$, define*

$$u(x) = \begin{cases} \displaystyle\int_{\partial B_R(0)} K(x, y)\, \varphi(y)\, d\sigma_y, & x \in B_R(0), \\ \varphi(x), & x \in \partial B_R(0). \end{cases}$$

Then u solves

$$\begin{cases} \Delta u = 0 & \text{in } B_R(0), \\ u = \varphi & \text{on } \partial B_R(0). \end{cases}$$

Proof. For $x \in B_R(0)$,

$$\Delta u(x) = \int_{\partial B_R(0)} \Delta_x K(x, y)\, \varphi(y)\, d\sigma_y.$$

Since $\Delta_x K(x, y) = 0$ for $x \in B_R(0)$ and $y \in \partial B_R(0)$ (the singularity of G lies at $y = x$, which is interior), and φ is bounded,

$$\int_{\partial B_R(0)} \Delta_x K(x, y)\, \varphi(y)\, d\sigma_y = 0.$$

It remains to check the boundary values. □

Theorem 3.5. *For every $x^0 \in \partial B_R(0)$,*

$$\lim_{B_R(0) \ni x \to x^0} u(x) = \varphi(x^0).$$

Proof. Fix $x^0 \in \partial B_R(0)$. Given $\varepsilon > 0$, choose $\delta > 0$ small enough that

$$|\varphi(y) - \varphi(x^0)| < \varepsilon \qquad \text{whenever } y \in \partial B_R(0) \text{ and } |y - x^0| < \delta.$$

Then, for $|x - x^0| < \delta/2$,

$$u(x) - \varphi(x^0) = \int_{\partial B_R(0)} K(x, y)\, \varphi(y)\, d\sigma_y - \varphi(x^0) = \int_{\partial B_R(0)} K(x, y)\big(\varphi(y) - \varphi(x^0)\big)\, d\sigma_y,$$

where we used $\int K = 1$. Splitting the integral,

$$\begin{aligned} \big|u(x) - \varphi(x^0)\big| \le &\int_{\partial B_R(0) \cap \{|y - x^0| < \delta\}} K(x, y)\, |\varphi(y) - \varphi(x^0)|\, d\sigma_y =: I_1 \\ &+ \int_{\partial B_R(0) \cap \{|y - x^0| \ge \delta\}} K(x, y)\, |\varphi(y) - \varphi(x^0)|\, d\sigma_y =: I_2. \end{aligned}$$

For the near region, $|I_1| \le \varepsilon$.

For the far region, $|x^0 - y| \ge \delta$ and $|x - x^0| < \delta/2$ imply

$$|y - x^0| \le |y - x| + \frac{\delta}{2} \le |y - x| + \frac{1}{2}|y - x^0|,$$

hence $|y - x^0| \le 2|y - x|$ (so y is not close to x or x^0). Thus

$$\begin{aligned} |I_2| &\le 2 \max|\varphi| \int_{\partial B_R(0) \cap \{|y - x^0| \ge \delta\}} K(x, y)\, d\sigma_y \\ &\le C \int_{\partial B_R(0) \cap \{|y - x^0| \ge \delta\}} \frac{R^2 - |x|^2}{R}\, |x - y|^{-n}\, d\sigma_y \\ &\le C(R^2 - |x|^2) \int_{\partial B_R(0) \cap \{|y - x^0| \ge \delta\}} |x^0 - y|^{-n}\, d\sigma_y \\ &\le C(R^2 - |x|^2)\, \delta^{-n}. \end{aligned}$$

As $x \to x^0$ one has $R^2 - |x|^2 \to 0$, so $I_2 \to 0$. Combining with $|I_1| \le \varepsilon$ yields the claim. □

Remark 3.6. In the Poisson integral formula, taking $x = 0$,

$$K(0, y) = \frac{1}{\omega_n} |y|^{-n} \frac{R^2}{R} = \frac{R}{\omega_n} |y|^{-n},$$

hence

$$u(0) = \int_{\partial B_R(0)} \frac{R}{\omega_n} |y|^{-n} \varphi(y)\, d\sigma_y = \frac{1}{R^{n-1}\omega_n} \int_{\partial B_R(0)} \varphi(y)\, d\sigma_y,$$

which is the mean value property.

Lemma 3.7 (Harnack's inequality in a ball)**.** *Suppose u is harmonic in $B_R(x_0)$ and $u \geq 0$. Then for $r = |x - x_0| < R$,*

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r}\, u(x_0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r}\, u(x_0).$$

Proof. Without loss of generality take $x_0 = 0$. The Poisson formula gives

$$u(x) = \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - |x|^2}{|x-y|^n}\, u(y)\, d\sigma_y.$$

For $|y| = R$ one has $R - |x| \leq |y - x| \leq R + |x|$, and therefore

$$\begin{aligned} \frac{1}{\omega_n R} \frac{R-|x|}{R+|x|} \left(\frac{1}{R+|x|}\right)^{n-2} \int_{\partial B_R} u(y)\, d\sigma_y &\leq u(x) \\ &\leq \frac{1}{\omega_n R} \frac{R+|x|}{R-|x|} \left(\frac{1}{R-|x|}\right)^{n-2} \int_{\partial B_R} u(y)\, d\sigma_y. \end{aligned}$$

By the mean value property,

$$\frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u(y)\, d\sigma_y = u(0),$$

whence

$$\frac{R-|x|}{R+|x|} \left(\frac{R}{R+|x|}\right)^{n-2} u(0) \leq u(x) \leq \frac{R+|x|}{R-|x|} \left(\frac{R}{R-|x|}\right)^{n-2} u(0).$$

□

Corollary 3.8 (Liouville)**.** *If $\Delta u = 0$ in $\mathbb{R}^n$ and u is bounded, then u is constant.*

Second approach. From Harnack's inequality,

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|}\, u(0) \leq u(x) \leq \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|}\, u(0).$$

Letting $R \to \infty$ yields $u(0) \leq u(x) \leq u(0)$, so $u \equiv C$. □

3.2 Green's function for a half-space

3.2.1 Half-space

Let

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) : x_n > 0\}.$$

Definition 3.9 (Reflection)**.** For $x \in \mathbb{R}^n_+$, write

$$x^* = (x_1, \ldots, x_{n-1}, -x_n) \in \mathbb{R}^n_-.$$

Idea. Take $\Phi(x,y) = -\Gamma(x^*,y)$. One needs

$$\begin{cases} \Delta_y \Phi(x,y) = 0, & y \in \mathbb{R}^n_+, \\ \Phi(x,y) = -\Gamma(x,y), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

The first holds because the only singularity is at $y = x^* \in \mathbb{R}^n_-$, while $y \in \mathbb{R}^n_+$. For the second: if $y \in \partial\mathbb{R}^n_+$, then $y = y^*$ and $|x-y| = |x^*-y|$, so

$$\Gamma(x^*,y) = \Gamma(x,y).$$

Thus $G(x,y) = \Gamma(x,y) - \Gamma(x^*,y)$ vanishes on the boundary (equivalently $G = \Gamma + \Phi$ with $\Phi = -\Gamma(x^*,\,\cdot\,)$).

Theorem 3.10 (Green's function for the half-space)**.**

$$G(x,y) = \begin{cases} \dfrac{1}{2\pi}\bigl(\log|x-y| - \log|x^*-y|\bigr), & n = 2, \\ -\dfrac{1}{(n-2)\omega_n}\bigl(|x-y|^{2-n} - |x^*-y|^{2-n}\bigr), & n \geq 3. \end{cases}$$

Lemma 3.11. *For $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$,*

$$\frac{\partial G}{\partial n}(x,y) = \frac{2x_n}{\omega_n}\,\frac{1}{|x-y|^n}.$$

Proof. Only the case $n \geq 3$ is written. One has $\partial G/\partial n = \nabla_y G \cdot \vec{n}$, and

$$\nabla_y G = \frac{1}{\omega_n}\left(|y-x|^{1-n}\frac{y-x}{|y-x|} - |y-x^*|^{1-n}\frac{y-x^*}{|y-x^*|}\right).$$

On the boundary $|y-x| = |y-x^*|$, so

$$\nabla_y G = \frac{1}{\omega_n}\,|y-x|^{-n}\,(y-x-y+x^*) = \frac{1}{\omega_n}\,|x-y|^{-n}\,(-x+x^*).$$

Here $-x+x^* = (0,\dots,-2x_n)$. Taking the outward unit normal $\vec{n} = (0,\dots,0,-1)$ gives

$$\nabla_y G \cdot \vec{n} = \frac{2x_n}{\omega_n\,|x-y|^n}.$$

$\square$

Definition 3.12 (Poisson kernel for the half-space)**.**

$$K(x,y) = \frac{\partial G}{\partial n}(x,y) = \frac{2x_n}{\omega_n\,|x-y|^n}$$

is the Poisson kernel for $\mathbb{R}^n_+$. Define

$$u(x) = \begin{cases} \displaystyle\int_{\partial\mathbb{R}^n_+} K(x,y)\,\varphi(y)\,d\sigma_y, & x \in \mathbb{R}^n_+, \\ \varphi(x), & x \in \partial\mathbb{R}^n_+. \end{cases}$$

Theorem 3.13. *Assume φ is bounded and continuous on $\partial\mathbb{R}^n_+$. Then:*

1. u solves

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = \varphi & \text{on } \partial\mathbb{R}^n_+; \end{cases}$$

2. $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$*;*

3. $\lim\limits_{\substack{x \to x^0 \\ x \in \mathbb{R}^n_+}} u(x) = \varphi(x^0)$ *for every* $x^0 \in \partial\mathbb{R}^n_+$.

Proof. (1) $\Delta_x u = \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y)\,\varphi(y)\,d\sigma_y = 0$, since $\Delta_x K(x,y) = 0$ for $x \in \mathbb{R}^n_+$ (the singularity would require $y = x$, which cannot occur for $y \in \partial\mathbb{R}^n_+$).

(2) Clear from the smoothness of K in $x \in \mathbb{R}^n_+$ and the boundedness of φ.

(3) Fix $x^0 \in \partial\mathbb{R}^n_+$. Given $\varepsilon > 0$, choose $\delta > 0$ so that $|\varphi(y) - \varphi(x^0)| < \varepsilon$ whenever $|y - x^0| < \delta$ and $y \in \partial\mathbb{R}^n_+$. Then

$$u(x) - \varphi(x^0) = \int_{\partial\mathbb{R}^n_+} K(x,y)\big(\varphi(y) - \varphi(x^0)\big)\,d\sigma_y,$$

and

$$\begin{aligned}
\big|u(x) - \varphi(x^0)\big| &\le \int_{\partial\mathbb{R}^n_+ \cap \{|y-x^0|<\delta\}} K(x,y)\,|\varphi(y) - \varphi(x^0)|\,d\sigma_y \\
&\quad + \int_{\partial\mathbb{R}^n_+ \cap \{|y-x^0|\ge\delta\}} K(x,y)\,|\varphi(y) - \varphi(x^0)|\,d\sigma_y \\
&\le C\varepsilon + (\max|\varphi|) \int_{\partial\mathbb{R}^n_+ \cap \{|y-x^0|\ge\delta\}} \frac{x_n}{\omega_n\,|x-y|^n}\,d\sigma_y \\
&\le C\varepsilon + C\,x_n \int_{\substack{\partial\mathbb{R}^n_+ \\ |y-x^0|\ge\delta}} |x-y|^{-n}\,d\sigma_y.
\end{aligned}$$

As $x_n \to 0$ (with $x \to x^0$), the second term tends to 0, so $|u(x) - \varphi(x^0)| \le C\varepsilon$ for x sufficiently close to x^0. $\square$

3.3 Perron's method

3.3.1 Perron's method

- We have shown existence of
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$
for the special domains $\Omega = B_R(0)$ and $\Omega = \mathbb{R}^n_+$, by writing out an explicit formula.
- This is *not* possible for a general domain Ω.
- We will introduce two approaches to show existence:
  - Perron's method;
  - energy method (calculus of variations).

3.3.2 Subharmonic functions

Definition 3.14 (Subharmonic / superharmonic)**.** A function $u \in C^2(\Omega)$ is *subharmonic* (respectively *superharmonic*) if $-\Delta u \le 0$ (respectively $-\Delta u \ge 0$) in Ω.

Theorem 3.15 (Equivalent characterizations)**.** *The following are equivalent:*

(i) u *is subharmonic:* $-\Delta u \le 0$ *whenever* Δu *exists;*

(ii) for every ball $B \subset\subset \Omega$ *and every* $h \in C^2(B) \cap C(\overline{B})$ *with* $\Delta h = 0$ *in* B*, if* $u \le h$ *on* ∂B *then* $u \le h$ *in* B*;*

(iii)

$$u(x) \le \frac{1}{|B_r|}\int_{B_r(x)} u(y)\,dy \qquad \forall\, r>0 \text{ with } B_r(x)\subset\Omega;$$

(iv)

$$u(x) \le \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,d\sigma_y \qquad \forall\, r>0 \text{ with } B_r(x)\subset\Omega.$$

Sketch. The equivalences (i) ⇔ (iii) ⇔ (iv) follow by the same arguments used for the mean value property of harmonic functions.

(ii) ⇐ (iii)/(iv). For every $x\in\Omega$ take $B_r(x)\subset\Omega$ with $u\le h$ on $\partial B_r(x)$. Then

$$u(x) \le \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,d\sigma_y \le \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} h(y)\,d\sigma_y = h(x).$$

(ii) ⇒ (iv). Take the special harmonic function h solving

$$\begin{cases} \Delta h = 0 & \text{in } B,\\ h\equiv u & \text{on } \partial B.\end{cases}$$

Then

$$h(x) = \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} h(y)\,d\sigma_y = \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,d\sigma_y,$$

and $u\le h$ on ∂B implies $u(x)\le h(x)$ by (ii). Hence

$$u(x) \le \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,d\sigma_y.$$

□

Lemma 3.16 (Maximum principle). *If u is subharmonic in Ω, then*

$$\sup_{\Omega} u = \max_{\partial\Omega} u,$$

or else $u\equiv C$.

Proof. Define

$$\Omega_1 := \left\{x : u(x) = \sup_{\Omega} u =: M\right\} \subset \Omega,$$

which is closed. The aim is to show that Ω_1 is open. For every $x\in\Omega_1$ there exists $B_\rho(x)\subset\Omega$ such that, by subharmonicity,

$$M = u(x) \le \frac{1}{|B_\rho|}\int_{B_\rho(x)} u(y)\,dy \le M.$$

Hence $u=M$ a.e. in B_ρ, so $B_\rho(x)\subset\Omega_1$. Therefore Ω_1 is either empty or equal to Ω:

- if $\Omega_1=\emptyset$, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$;
- if $\Omega_1=\Omega$, then $u\equiv M$ in Ω.

□

Lemma 3.17. *Let u be subharmonic and v superharmonic, with $u-v\le 0$ on $\partial\Omega$. Then either $u<v$ in Ω, or $u\equiv v$ in Ω.*

Proof. The difference $u-v$ is subharmonic. □

Remark 3.18. A harmonic function is both subharmonic and superharmonic.

3.3.3 Harmonic lifting

Let u be subharmonic in Ω, let $B \subset\subset \Omega$ be a fixed ball, and let $\bar{u}$ be harmonic in B with $\bar{u} = u$ on ∂B. Then (using the Green's function for the ball) either $u < \bar{u}$ in B, or $u \equiv \bar{u}$ in B.

Definition 3.19 (Harmonic lifting)**.** The function

$$U(x) = \begin{cases} \bar{u}(x), & x \in B, \\ u(x), & x \in \Omega \setminus B, \end{cases}$$

is called a *harmonic lifting* of u in B.

Lemma 3.20. *The lifting U is subharmonic in Ω.*

Proof. Let $B' \subset\subset \Omega$ and let h be harmonic with $h \geq U$ on $\partial B'$. We need to show $U \leq h$ in B'. Write

$$\begin{aligned} C_1 &:= \partial B' \setminus B, \\ C_3 &:= \partial B' \cap B, \\ C_2 &:= \partial B \cap B'. \end{aligned}$$

Note that $\Delta h = 0$ and $\Delta \bar{u} = 0$.

On $C_1 = \partial B' \setminus B$ one has $h \geq U = u$ by assumption (on the whole of $\partial B'$). On $C_3 = \partial B' \cap B$ one has $h \geq U = \bar{u} \geq u$. Hence $h \geq u$ on $\partial B'$, and since h is harmonic and u is subharmonic,

$$h \geq u \quad \text{in } B'.$$

In particular, in $B' \setminus B$ we have $h \geq u = U$.

It remains to show $h \geq U$ in $B' \cap B$. On $C_2 = \partial B \cap B'$ one has $u = \bar{u} = U$, so $h \geq \bar{u}$. On $C_3 = \partial B' \cap B$ one has $h \geq U = \bar{u}$. Since both h and $\bar{u}$ are harmonic,

$$h \geq \bar{u} \quad \text{in } B' \cap B.$$

Thus $h \geq U$ on B', and therefore U is subharmonic. $\square$

Lemma 3.21. *If $u_1, \ldots, u_N$ are subharmonic in Ω, then so is* $\max\{u_1, \ldots, u_N\}$.

4 Perronś method and barriers

4.1 Equivalent characterizations of subharmonic functions

Recall the following are equivalent:

(i) $-\Delta u \leq 0$ (if the Laplacian exists);

(ii) for every ball $B \subset\subset \Omega$ and every harmonic h with $u \leq h$ on ∂B, one has $u \leq h$ in B;

(iii) $u(x) \leq \dfrac{1}{|B_r|} \displaystyle\int_{B_r(x)} u(y)\, dy$;

(iv) $u(x) \leq \dfrac{1}{|\partial B_r|} \displaystyle\int_{\partial B_r(x)} u(y)\, d\sigma_y$.

Proof of (i)⇔(iii)/(iv). Define the spherical mean

$$\phi(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\, d\sigma_y.$$

With the change of variables $y - x = rz$, $|z| = 1$, so that $d\sigma_y = r^{n-1}\, d\sigma_z$,

$$\phi(r) = \frac{1}{\omega_n r^{n-1}}\, r^{n-1} \int_{|z|=1} u(x + rz)\, d\sigma_z = \frac{1}{\omega_n} \int_{|z|=1} u(x + rz)\, d\sigma_z.$$

Differentiating in r,

$$\phi'(r) = \frac{1}{\omega_n} \int_{|z|=1} z \cdot \nabla u(x + rz)\, d\sigma_z = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r} \frac{y - x}{r} \cdot \nabla u\, d\sigma_y = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r} \frac{\partial u}{\partial n}\, d\sigma_y = \frac{1}{\omega_n r^{n-1}} \int_{B_r} \Delta u\, dy \geq$$

where the last step uses the divergence theorem and $-\Delta u \le 0$ (i.e. $\Delta u \ge 0$). Hence $\phi(r) \ge \phi(0)$, that is

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} u(y)\, d\sigma_y \ge u(x).$$

Integrating in r yields the volume mean inequality

$$u(x) \le \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy.$$

□

Proof of (iii)/(iv)⇒(ii). For every $x \in \Omega$ and every ball $B_r(x) \subset\subset \Omega$, if $u \le h$ on $\partial B_r(x)$ with h harmonic, then

$$u(x) \le \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y \le \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} h(y)\, d\sigma_y = h(x).$$

□

Proof of (ii)⇒(iii)/(iv). For every $x \in \Omega$ and every $B_r(x) \subset\subset \Omega$, define h by

$$\begin{cases} \Delta h = 0 & \text{in } B_r(x), \\ h = u & \text{on } \partial B_r(x), \end{cases}$$

via the Green's function. Then by (ii),

$$u(x) \le h(x) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} h\, d\sigma = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u\, d\sigma,$$

and likewise for the volume mean of the harmonic function h. □

4.2 Harmonic lifting and the idea of Perron's method

Harmonic lifting on a ball B. Define

$$U = \begin{cases} \bar{u} & x \in B, \\ u & x \in B^c, \end{cases}$$

where $\bar{u}$ is harmonic in B and $\bar{u} \equiv u$ on ∂B.

In one dimension, subharmonicity means $-u'' \le 0$ (i.e. u is convex). On an interval $[a, b]$ the harmonic lifting is the linear interpolant

$$\bar{u}(x) = \frac{(x - a)u(b) + (b - x)u(a)}{b - a}.$$

Idea of Perron's method (in 1D). Consider

$$\begin{cases} -u'' = 0, \\ u(0) = u_0, \quad u(1) = u_1. \end{cases}$$

The exact solution is

$$u = u_0(1-x) + u_1 x.$$

Starting from $v \in \mathcal{S}_\varphi$, one repeatedly replaces v on subintervals by its harmonic lifting; continually lifting subharmonic functions pushes them toward the solution.

4.3 Existence via Perron's method

We turn to existence of a solution to

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Let

$$\mathcal{S}_\varphi = \big\{ v \text{ subharmonic in } \Omega : \ v \le \varphi \text{ on } \partial\Omega \big\}.$$

Theorem 4.1. *The function*

$$u \equiv \sup_{v \in \mathcal{S}_\varphi} v$$

is harmonic in Ω*.*

Proof. **Step 1:** u is well-defined.

- $\mathcal{S}_\varphi \neq \emptyset$: the constant $V \equiv \min_{\partial\Omega} \varphi$ is subharmonic.
- $u \not\equiv +\infty$: for every $v \in \mathcal{S}_\varphi$,

$$v(x) \le \sup_{\partial\Omega} \varphi \qquad \forall x \in \Omega.$$

In particular,

$$\inf_{\partial\Omega} \varphi \le u(x) \le \sup_{\partial\Omega} \varphi.$$

Step 2: Fix any $x \in \Omega$. There exists a sequence $\{v_n\} \subset \mathcal{S}_\varphi$ with $\lim_{n\to\infty} v_n(x) = u(x)$. Replacing v_k by $\max\{v_1, \dots, v_k, \inf \varphi\}$, we may assume $\{v_n\}$ is increasing and bounded from below. Let $B = B_R(x) \subset\subset \Omega$, and let $\overline{V}_n$ be the harmonic lifting of v_n in $B_R(x)$. Then $\overline{V}_n \in \mathcal{S}_\varphi$ and

$$\lim_{n\to\infty} \overline{V}_n(x) = u(x). \tag{$*$}$$

Proof of $(*)$. By definition of lifting, $v_n(x) \le \overline{V}_n(x)$, so $u(x) \le \lim_n \overline{V}_n(x)$. On the other hand, $\overline{V}_n \in \mathcal{S}_\varphi$ implies $\overline{V}_n(x) \le \sup_{v \in \mathcal{S}_\varphi} v = u(x)$, hence

$$u(x) \le \lim_n \overline{V}_n(x) \le u(x).$$

□

Step 3: The limit $v := \lim_{n\to\infty} \overline{V}_n$ is harmonic in $B_{R/2}(x)$. (Here $\{\overline{V}_n\}$ is harmonic in $B_R(x)$.) By the gradient estimate, for any harmonic h,

$$\sup_{B_{R/2}(x)} |D^\alpha h| \le C \sup_{B_R(x)} |h|.$$

Hence

$$\sup_n \sup_{B_{R/2}(x)} |D\overline{V}_n| \le C.$$

A subsequence $\overline{V}_{n_k}$ therefore converges uniformly on $B_{R/2}(x)$ to v. By the MVP (uniform convergence is needed), v is harmonic.

Step 4: We show $u \equiv v$ in $B_{R/2}(x)$ (note $v \le u$). Assume there exists $x' \in B_{R/2}(x)$ with $u(x') > v(x')$. Then there exists $u_0 \in \mathcal{S}_\varphi$ with

$$v(x') < u_0(x') < u(x').$$

Set $w_k := \max(u_0, \overline{V}_{n_k}) \ge u_0$, and let $\overline{w}_k$ be the harmonic lifting of w_k in $B_R(x)$. Then $\overline{w}_k$ converges uniformly on $B_{R/2}(x)$ to a harmonic function w, and

$$v(y) \le w(y) \le u(y) \qquad \forall y \in B_{R/2}(x).$$

Evaluating at the center and using ($*$) gives $v(x) = w(x) = u(x)$. By the MVP,

$$w(x) = \frac{1}{|B_{R/2}(x)|} \int_{B_{R/2}(x)} w(y)\,dy \ge \frac{1}{|B_{R/2}(x)|} \int_{B_{R/2}(x)} v(y)\,dy = v(x) = w(x),$$

so $v \equiv w$ in $B_{R/2}(x)$. This contradicts $w(x') \ge u_0(x') > v(x')$. Hence $u \equiv v$ in $B_{R/2}(x)$, and u is harmonic in $B_{R/2}(x)$.

Step 5: Since $x \in \Omega$ is arbitrary, u is harmonic in Ω. $\square$

4.4 Boundary behavior and barriers

Boundary behavior. Does $u = \varphi$ on $\partial\Omega$?

Definition 4.2 (Barrier)**.** Let $\xi \in \partial\Omega$. A function $\beta \in C(\overline{\Omega})$ is called a *barrier* at ξ with respect to Ω if

i) $\beta > 0$ in $\overline{\Omega} \setminus \{\xi\}$, and $\beta(\xi) = 0$;

ii) β is superharmonic in Ω.

Definition 4.3 (Regular point)**.** A point $\xi \in \partial\Omega$ is *regular* if there exists a barrier β at ξ with respect to Ω.

Remark 4.4. A barrier is a local property at $\partial\Omega$. If β is a barrier at ξ with respect to Ω, then for every ball $B_\rho(\xi)$ one may form

$$\tilde{\beta} \equiv \begin{cases} m \equiv \inf_{\Omega \setminus B_\rho} \beta & x \in \overline{\Omega} \setminus B_\rho, \\ \min(m, \beta) & x \in \overline{\Omega} \cap B_\rho, \end{cases}$$

which is again a barrier.

Theorem 4.5. *Let $u \equiv \sup_{v \in \mathcal{S}_\varphi} v$ be as in Theorem 4.1. If ξ is a regular point of $\partial\Omega$ and φ is continuous at ξ, then*

$$\lim_{\Omega \ni x \to \xi} u(x) = \varphi(\xi).$$

Proof. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(\xi)| < \varepsilon \qquad \text{if } x \in \partial\Omega \text{ and } |x - \xi| < \delta.$$

Let $M = \sup_{\partial\Omega} |\varphi|$, and let β be a barrier at ξ. Define $g : \partial\Omega \to \mathbb{R}$ by

$$g(x) = \liminf_{\substack{y \to x \\ y \in \Omega}} \beta(y).$$

Then g attains its minimum m_1 on $\{|x - \xi| \ge \delta\} \cap \partial\Omega$. Define

$$\overline{V}(y) = \varphi(\xi) + \varepsilon + \frac{\beta(y)}{m_1}\big(2M - \varphi(\xi)\big).$$

(Note $\beta(\xi) = 0$ implies $\overline{V}(\xi) = \varphi(\xi) + \varepsilon$.) Then $\overline{V}$ is superharmonic since β is.

Boundary values of $\overline{V}$.

- If $x \in \partial\Omega$ and $|x-\xi| < \delta$, then

$$\liminf_{\substack{y\to x\\ y\in\Omega}} \overline{V}(y) \ge \varphi(\xi) + \varepsilon \ge \varphi(x).$$

- If $x \in \partial\Omega$ and $|x-\xi| \ge \delta$, then

$$\liminf_{\substack{y\to x\\ y\in\Omega}} \overline{V}(y) \ge \varphi(\xi) + \varepsilon + 2M - \varphi(\xi) = 2M + \varepsilon > \varphi(x).$$

Thus $\overline{V}$ is superharmonic and $\overline{V} \ge \varphi$ on $\partial\Omega$. Recalling that u is harmonic with $u \le \varphi$ on $\partial\Omega$, the strong maximum principle yields $u \le \overline{V}$ in Ω, and

$$\limsup_{\substack{y\to \xi\\ y\in\Omega}} u(y) \le \overline{V}(\xi) = \varphi(\xi) + \varepsilon.$$

Since ε is arbitrary,

$$\limsup_{\substack{y\to \xi\\ y\in\Omega}} u(y) \le \varphi(\xi).$$

Similarly, define

$$\underline{V}(y) = \varphi(\xi) - \varepsilon - \frac{\beta(y)}{m_2}\big(\varphi(\xi) + 2M\big),$$

where $\limsup_{y\to x,\, y\in\Omega} \beta \le m_2$ for $|x-\xi| \ge \delta$. One can show

$$\begin{cases} \underline{V} \text{ is subharmonic,} \\ \underline{V} \le \varphi \text{ on } \partial\Omega, \end{cases}$$

i.e. $\underline{V} \in \mathcal{S}_\varphi$, whence $\underline{V} \le u$. Therefore

$$\varphi(\xi) - \varepsilon \le \liminf_{\substack{y\to \xi\\ y\in\Omega}} \underline{V}(y) \le \liminf_{\substack{y\to \xi\\ y\in\Omega}} u(y).$$

Combining the liminf and limsup estimates,

$$\varphi(\xi) \le \liminf_{\substack{y\to \xi\\ y\in\Omega}} u(y) \le \limsup_{\substack{y\to \xi\\ y\in\Omega}} u(y) \le \varphi(\xi),$$

so

$$\lim_{\substack{y\to \xi\\ y\in\Omega}} u(y) = \varphi(\xi).$$

□

Examples of barriers: exterior sphere condition. Suppose that for every $\xi \in \partial\Omega$ there exist $R > 0$ and $y \in \mathbb{R}^n$ such that

$$B_R(y) \cap \overline{\Omega} = \{\xi\}.$$

Define

$$\beta(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & n \ge 3, \\ \log\big(|x-y|/R\big) & n = 2. \end{cases}$$

Then:

- $\beta(x) > 0$ for all $x \in \overline{\Omega} \setminus \{\xi\}$;
- $\Delta_x \beta \le 0$ in Ω (superharmonic);
- $\beta(\xi) = 0$.

Hence β is a barrier at ξ.

5 Dirichlet principle

Perron's method is not a convenient numerical device. An alternative is the *energy method*, which recasts the Dirichlet problem as a minimization of the Dirichlet integral.

5.1 Energy uniqueness

Consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{$*$}$$

Previously uniqueness followed from the maximum principle. Now assume u_1, u_2 are two solutions and set $w := u_1 - u_2$. Then

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Green's identity gives

$$0 = \int_\Omega w\,\Delta w\,dx = \int_{\partial\Omega} w\frac{\partial w}{\partial n}\,d\sigma - \int_\Omega \nabla w\cdot\nabla w\,dx.$$

The boundary integral vanishes since $w = 0$ on $\partial\Omega$, so

$$0 = \int_\Omega |\nabla w|^2\,dx \geq 0 \implies \nabla w \equiv 0 \text{ in } \Omega \implies w \equiv C \text{ in } \Omega.$$

But $w = 0$ on $\partial\Omega$, hence $w \equiv 0$ in $\overline{\Omega}$, i.e. $u_1 \equiv u_2$.

5.2 The energy functional

Define

$$E[v] := \int_\Omega \Big(\tfrac{1}{2}|\nabla v|^2 - fv\Big)\,dx,$$

on the admissible class

$$\Lambda = \big\{ v \in C^2(\Omega)\cap C(\overline{\Omega}) \;\big|\; v = g \text{ on } \partial\Omega \big\}.$$

Theorem 5.1 (Dirichlet's principle)**.** *1. If $u \in \Lambda$ solves ($*$), then*

$$E[u] \leq E[v] \qquad \forall v \in \Lambda.$$

2. If $u \in \Lambda$ satisfies $E[u] \leq E[v]$ for all $v \in \Lambda$, then u solves ($*$).
In other words,

$$u \in \operatorname{argmin}_{v\in\Lambda} E[v], \qquad \textit{or} \qquad \min_{v\in\Lambda} E[v] = E[u].$$

Proof. (1) For every $v \in \Lambda$,

$$\begin{aligned} 0 &= \int_\Omega (-\Delta u - f)(u - v)\,dx && \big(u - v|_{\partial\Omega} = 0\big) \\ &= \int_\Omega \nabla u\cdot\nabla(u-v) - f(u-v)\,dx. \end{aligned}$$

Hence

$$\int_\Omega \big(|\nabla u|^2 - uf\big)\,dx = \int_\Omega \big(\nabla u\cdot\nabla v - fv\big)\,dx \leq \int_\Omega \Big(\tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}|\nabla v|^2 - fv\Big)\,dx,$$

where we used $\nabla u\cdot\nabla v \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2$. Rearranging yields

$$E[u] = \int_\Omega \Big(\tfrac{1}{2}|\nabla u|^2 - uf\Big)\,dx \leq \int_\Omega \Big(\tfrac{1}{2}|\nabla v|^2 - fv\Big)\,dx = E[v] \qquad \forall v \in \Lambda.$$

(2) Conversely, fix any $\varphi \in C_c^\infty(\Omega)$. Denote $F(s) := E[u+s\varphi]$ for $s \in \mathbb{R}$. Since $\varphi \in C_c^\infty(\Omega)$ one has $\varphi|_{\partial\Omega} = 0$, so $u + s\varphi \in \Lambda$. Minimality gives $E[u] \le E[u+s\varphi]$, i.e.

$$\begin{aligned}
&\int_\Omega \tfrac{1}{2}\big(|\nabla u|^2 - |\nabla(u+s\varphi)|^2\big) - f\big(u-(u+s\varphi)\big)\,dx \le 0\\
&\implies \int_\Omega \tfrac{1}{2}\big(|\nabla u|^2 - |\nabla u|^2 - s^2|\nabla\varphi|^2 - 2s\nabla u\cdot\nabla\varphi\big) + sf\varphi\,dx \le 0\\
&\implies \int_\Omega -\tfrac{s^2}{2}|\nabla\varphi|^2 + s\varphi(\Delta u + f)\,dx \le 0\\
&\implies \int_\Omega \varphi(\Delta u + f)\,dx \le \int_\Omega \frac{|\nabla\varphi|^2}{2}\,s \qquad \forall s > 0.
\end{aligned}$$

Letting $s \to 0^+$ (and repeating with $-\varphi$) yields $-\Delta u = f$ in Ω.

Alternatively, $F'(0) = 0$ gives

$$\begin{aligned}
0 = \lim_{s\to 0}\frac{F(s)-F(0)}{s} &= \lim_{s\to 0}\frac{1}{s}\int_\Omega \Big(\tfrac{s^2}{2}|\nabla\varphi|^2 + s\nabla u\cdot\nabla\varphi - sf\varphi\Big)\,dx\\
&= \int_\Omega \big(\nabla u\cdot\nabla\varphi - f\varphi\big)\,dx = \int_\Omega \varphi(-\Delta u - f)\,dx.
\end{aligned}$$

Since this holds for all $\varphi \in C_c^\infty(\Omega)$, we conclude $-\Delta u = f$ in Ω. $\square$

5.3 Convexity and the direct method

Remark 5.2 (Convexity of E)**.** The energy E is convex, i.e. for $\lambda \in [0,1]$,

$$E\big[\lambda u + (1-\lambda)v\big] \le \lambda E[u] + (1-\lambda)E[v].$$

One needs to check that $\lambda u + (1-\lambda)v \in \Lambda$ whenever $u, v \in \Lambda$; this holds because

$$\big(\lambda u + (1-\lambda)v\big)\Big|_{\partial\Omega} = \lambda g + (1-\lambda)g = g.$$

To see convexity, expand

$$\begin{aligned}
E\big[\lambda u + (1-\lambda)v\big] &= \int_\Omega \Big(\tfrac{1}{2}\big|\lambda\nabla u + (1-\lambda)\nabla v\big|^2 - f\big(\lambda u + (1-\lambda)v\big)\Big)\,dx\\
&= \int_\Omega \Big(\tfrac{1}{2}\lambda^2|\nabla u|^2 + \tfrac{1}{2}(1-\lambda)^2|\nabla v|^2 + \lambda(1-\lambda)\,\nabla u\cdot\nabla v - \lambda f u - (1-\lambda)fv\Big)\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&E\big[\lambda u + (1-\lambda)v\big] - \lambda E[u] - (1-\lambda)E[v]\\
&\quad= \int_\Omega \Big(\tfrac{1}{2}(\lambda^2-\lambda)\,|\nabla u|^2 + \tfrac{1}{2}\big((1-\lambda)^2 - (1-\lambda)\big)\,|\nabla v|^2 + \lambda(1-\lambda)\,\nabla u\cdot\nabla v\Big)\,dx\\
&\quad= -\lambda(1-\lambda)\int_\Omega \Big(\tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}|\nabla v|^2 - \nabla u\cdot\nabla v\Big)\,dx\\
&\quad= -\lambda(1-\lambda)\int_\Omega \tfrac{1}{2}|\nabla u - \nabla v|^2\,dx \le 0.
\end{aligned}$$

Outlook. Does E have a minimizer?

1. Later one shows that E has a minimizer in a larger space — a *Sobolev space*.
2. One then shows that the minimizer lies in Λ again (*regularity theory*).

6 Finite difference methods for elliptic equations

The Dirichlet principle produces a well-posed Poisson problem, but does not by itself yield a computable solution. On a bounded interval (or a rectangle) the Laplacian may be replaced by a second-order central difference, leading to a sparse linear system whose stability mirrors the continuous maximum principle.

6.1 Interpolation and the central difference

Remark 6.1 (Taylor / piecewise linear)**.** $I_1(x) = f(a) + f'(a)(x-a)$ satisfies

$$f(x) - I_1(x) = \frac{f''(\xi)}{2}(x-a)^2 \le \frac{\|f''\|_\infty}{2} h^2,$$

while the constant interpolant I_0 has error $O(h)$.

On $[a,b]$ with $b = a + Nh$, use values of f at the $N+1$ nodes $a, a+h, \dots, b$. Advantages of piecewise linear interpolation: derivatives of f are not required; the approximant is continuous; second-order accurate.

On the i-th subinterval $[a+(i-1)h, a+ih]$,

$$\begin{aligned} I_1(x) &= f\big(a+(i-1)h\big)\frac{a+ih-x}{h} + f(a+ih)\frac{x-\big(a+(i-1)h\big)}{h}, \\ I_0(x) &= f\big(a+(i-1)h\big) \qquad \text{(discontinuous)}. \end{aligned}$$

Hat-function form:

$$\widetilde{I}_h(x) = \sum_j f(x_j)\,\phi_j(x),$$

where ϕ_j is the standard piecewise-linear "hat" supported on $[x_{j-1}, x_{j+1}]$ (uses information from 3 points, more accurate, but still second-order $O(h^2)$). The same hats span the P_1 finite-element space in Section 19.

6.2 Model BVP and L^∞ stability

Consider the 1D Poisson problem

$$\begin{cases} -u_{xx}(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad f \in C. \tag{14}$$

(Here Δ encodes diffusion/viscosity.) Existence and uniqueness hold; if $f \in C$ then $u \in C^2$; if $f \ge 0$ then $u \ge 0$.

Remark 6.2 (L^∞ estimate)**.** For $-\Delta u = f$ in $\Omega \subseteq \mathbb{R}^{2,3}$ with $u|_{\partial\Omega} = 0$,

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}$$

(zero boundary data, so u cannot be a nonzero constant). In the 1D model on $(0,1)$,

$$\|u\|_{L^\infty(0,1)} \le \frac{1}{8}\|f\|_{L^\infty(0,1)}.$$

Exercise 6.3. For $-u_{xx} = f$ on $(0,1)$ with $u(0) = u(1) = 0$, prove $\|u\|_{L^\infty} \le \frac{1}{8}\|f\|_{L^\infty}$. Hint: consider $V = u + \frac{1}{2}\big(x - \frac{1}{2}\big)^2\|f\|_\infty$.

6.3 Central differences and the discrete matrix

Return to (14). Take $h = \frac{1}{N}$, partition the interval into N subintervals, $x_j = jh$, $j = 0, \dots, N$, and write $\hat{u}_j \approx u_j \equiv u(x_j)$.

On $[x_j, x_{j+1}]$, the linear approximation

$$I(x) = u(x_j)\frac{x_{j+1} - x}{h} + u(x_{j+1})\frac{x - x_j}{h}, \qquad I'(x) = \frac{u(x_{j+1}) - u(x_j)}{h} \approx u'(x).$$

For the second derivative,

$$\begin{aligned} v''(x_j) &\approx \frac{1}{h}\Big(v'\big(x_j + \tfrac{h}{2}\big) - v'\big(x_j - \tfrac{h}{2}\big)\Big) \\ &\approx \frac{1}{h}\left(\frac{v(x_{j+1}) - v(x_j)}{h} - \frac{v(x_j) - v(x_{j-1})}{h}\right) = \frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{h^2}. \end{aligned}$$

Hence $-\partial_{xx} u = f$ becomes, for $j = 1, \dots, N-1$,

$$-\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} = f(x_j), \tag{15}$$

with boundary conditions $u(x_0) = u(x_N) = 0$.

In matrix form one may write a discrete system for the full vector $\underline{u} = (u_0, \dots, u_N)^T$, but the natural way to impose Dirichlet data $u_0 = u_N = 0$ is to eliminate the boundary unknowns and solve $B\underline{\widetilde{u}} = \underline{f}$ for $\underline{\widetilde{u}} = (u_1, \dots, u_{N-1})^T$, where

$$B = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \qquad \underline{f} = \big(f(x_1), \dots, f(x_{N-1})\big)^T.$$

The matrix B is symmetric positive definite: for $v \in \mathbb{R}^{N-1}$,

$$v^T B v = \frac{1}{h^2}\left(v_1^2 + \sum_{i=1}^{N-2}(v_i - v_{i+1})^2 + v_{N-1}^2\right) \geq 0,$$

with equality iff $v \equiv 0$. Hence B is nonsingular and $\underline{\widetilde{u}} = B^{-1}\underline{f}$ exists and is unique.

Remark 6.4. Enlarging to an $(N+1) \times (N+1)$ stencil whose first/last rows are $\frac{1}{h^2}(1, -1, 0, \dots)$ and $\frac{1}{h^2}(\dots, 0, -1, 1)$ produces a *singular* Neumann matrix (constants lie in the kernel). Dirichlet conditions must be enforced by eliminating u_0, u_N as above, not by that enlarged stencil.

6.4 Error analysis (qualitative setup)

1. How different are the properties of the discrete solution formed by $\underline{u}$ from those of u?

2. How accurate as $h \to 0$?

3. $|u(x_j) - \hat{u}_j| = ?$

Model test problem:

$$\begin{cases} -\partial_{xx} u = \sin(\pi x), \\ u(0) = u(1) = 0. \end{cases} \tag{16}$$

6.5 Discrete matrix B: non-negativity and stability

Return to the 1D Dirichlet Poisson problem

$$\begin{cases} -\partial_{xx} u = f & \text{in } (0, L), \\ u(0) = u(L) = 0, \end{cases}$$

with $h = L/(n+1)$ and the reduced system $B\underline{u} = \underline{f}$,

$$B = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}, \qquad \underline{u} = B^{-1}\underline{f}.$$

The continuous problem satisfies

(a) non-negativity: $f \ge 0 \Rightarrow u \ge 0$;

(b) boundedness: $\|u\|_\infty \le \frac{1}{8}\|f\|_\infty$ (on $(0,1)$).

Proposition 6.5 (Discrete non-negativity). *Write $B^{-1} = (b_{ij})$. Then $b_{ij} \ge 0$ for all $1 \le i, j \le n$. Hence if $f_j \ge 0$ for all j,*

$$u_i = \sum_{j=1}^{n} b_{ij} f_j \ge 0.$$

Proposition 6.6 (Discrete stability). *$0 \le \sum_{j=1}^{n} b_{ij} \le \frac{1}{8}$ for every $1 \le i \le n$. Consequently*

$$\|\underline{u}\|_\infty = \max_i \Big|\sum_j b_{ij} f_j\Big| \le \max_i \Big(\sum_j b_{ij}\Big) \|\underline{f}\|_\infty \le \tfrac{1}{8}\|\underline{f}\|_\infty,$$

i.e. $\|B^{-1}\|_\infty \le \frac{1}{8}$.

Remark 6.7 (Discrete norms). For $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$,

$$\|v\|_\infty = \max_{1\le j\le n} |v_j|, \qquad \|v\|_2 = \Big(\sum_{j=1}^{n} |v_j|^2\Big)^{1/2},$$

and the induced matrix norms are

$$\|A\|_p = \sup_{v\ne 0} \frac{\|Av\|_p}{\|v\|_p}, \qquad \|A\|_\infty = \max_i \sum_j |a_{ij}|, \qquad \|A\|_1 = \max_j \sum_i |a_{ij}|, \qquad \|A\|_2 = \sqrt{\lambda_{\max}(A^* A)}.$$

All finite-dimensional norms are equivalent: there exist $C_1, C_2 > 0$ with $C_1\|A\|_p \le \|A\|_q \le C_2\|A\|_p$. The discrete L^2 norm of a grid function is $\big(\sum_i |f(x_i)|^2 \Delta x\big)^{1/2}$, approximating $\big(\int_a^b |f|^2\,dx\big)^{1/2}$.

6.6 Truncation error, consistency, and convergence

Assume $v \in C^4$. Taylor expansion yields the central-difference identity

$$\frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{h^2} = v''(x_j) + \frac{h^2}{12} v^{(4)}(\xi)$$

for some $\xi \in [x_{j-1}, x_{j+1}]$. Applied to $-\partial_{xx} u = f$,

$$\frac{-u(x_{j+1}) + 2u(x_j) - u(x_{j-1})}{h^2} = f(x_j) + \tau_j, \qquad \tau_j := -\frac{h^2}{12} u^{(4)}(\xi_j)$$

(τ_j = truncation error).

Definition 6.8 (Discretization error). $e_j := u(x_j) - u_j$, where u_j is the numerical solution (discretization error).

Comparing the exact grid values (including truncation) with the numerical scheme gives

$$\frac{-e_{j+1} + 2e_j - e_{j-1}}{h^2} = \tau_j, \qquad 1 \le j \le n, \qquad e_0 = e_{n+1} = 0,$$

or in matrix form

$$B\underline{e} = \underline{\tau}, \qquad \underline{\tau} = \frac{h^2}{12}\big(u^{(4)}(\xi_1), \ldots, u^{(4)}(\xi_n)\big)^T.$$

Theorem 6.9 (A priori error estimate). *Using discrete stability $\|B^{-1}\|_\infty \le \frac{1}{8}$,*

$$\|\underline{e}\|_\infty \le \tfrac{1}{8}\|\underline{\tau}\|_\infty \le \frac{h^2}{96} \max_{0 \le x \le 1} |u^{(4)}(x)|.$$

Thus the scheme is second-order, $O(h^2)$, provided $u \in C^4$.

Definition 6.10 (Consistency). Consistency: $\lim\limits_{h \to 0} \|\underline{\tau}\|_\infty = 0$.

For a general elliptic discretization $B\underline{u} = \underline{f}$ and the consistency relation $B\underline{\hat{u}} = \underline{f} + \underline{\tau}$, the error $\underline{e} = \underline{\hat{u}} - \underline{u}$ satisfies the *error equation*

$$B\underline{e} = \underline{\tau} \qquad \Rightarrow \qquad \underline{e} = B^{-1}\underline{\tau}.$$

Stability $\|B^{-1}\|_\infty \le C$ then yields $\|\underline{e}\|_\infty \le C\|\underline{\tau}\|_\infty \to 0$ as $h \to 0$:

$$\textbf{stability} + \textbf{consistency} \;=\; \textbf{convergence}.$$

In the 1D Dirichlet Poisson case, $\|\underline{\tau}\|_\infty \le C|h|^2$ (second-order).

6.7 Neumann and Robin boundary conditions

6.7.1 Pure Neumann is not unique

The continuous problem

$$\begin{cases} -\partial_{xx} u = f & \text{in } (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

is not unique: if u_0 is a solution then so is $u_0 + C$. A well-posed model is the reaction–diffusion BVP with $c > 0$,

$$\begin{cases} -\partial_{xx} u + cu = f, \\ u'(0) = \alpha, \quad u'(1) = \beta. \end{cases}$$

6.7.2 Interior stencil

With $x_i = ih$, $x_0 = 0$, $x_N = 1$, the interior equations are

$$\frac{-u_{i-1} + 2u_i - u_{i+1} + ch^2 u_i}{h^2} = f_i, \qquad i = 1, \ldots, N-1.$$

6.7.3 Method 1: one-sided differences (first-order)

At $x = 0$, $\frac{u_1 - u_0}{h} \approx \alpha$, rewritten as $\frac{1}{h^2}(u_0 - u_1) = -\frac{\alpha}{h}$. At $x = 1$, $\frac{u_N - u_{N-1}}{h} \approx \beta$, rewritten as $\frac{1}{h^2}(u_N - u_{N-1}) = \frac{\beta}{h}$. The resulting $(N+1) \times (N+1)$ system has diagonal entries > 0, but the boundary approximation is only first-order:

$$\frac{u_1 - u_0}{h} - u'(0) = O(h) \qquad \text{(first-order accuracy)}.$$

6.7.4 Method 2: ghost points (second-order)

Introduce a ghost value u_{-1}. At x_0,

$$-u_{-1} + (2 + ch^2)u_0 - u_1 = f_0 h^2,$$
$$\frac{u_1 - u_{-1}}{2h} \approx \alpha \qquad \Rightarrow \qquad u_{-1} = u_1 - 2\alpha h.$$

Eliminating u_{-1} yields

$$(2 + ch^2)u_0 - 2u_1 = f_0 h^2 - 2\alpha h.$$

Similarly at x_N with ghost $u_{N+1} = u_{N-1} + 2\beta h$,

$$-2u_{N-1} + (2 + ch^2)u_N = h^2 f_N + 2\beta h.$$

The matrix equation is

$$\frac{1}{h^2}\begin{pmatrix} 2+ch^2 & -2 & & & \\ -1 & 2+ch^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+ch^2 & -1 \\ & & & -2 & 2+ch^2 \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} f_0 - \dfrac{2\alpha}{h} \\ f_1 \\ \vdots \\ f_{N-1} \\ f_N + \dfrac{2\beta}{h} \end{pmatrix}.$$

Taylor expansion confirms second-order accuracy of the centered Neumann formula:

$$\frac{u(h) - u(-h)}{2h} = u'(0) + O(h^2) \qquad \text{(second-order accuracy!!)}.$$

6.7.5 Robin / mixed BVP

Exercise 6.11 (Robin)**.**

$$\begin{cases} -\partial_{xx} u = f, \\ u(0) = \alpha, \quad u'(1) = \beta \end{cases}$$

combines Dirichlet and Neumann (homework).

6.8 2D Dirichlet Poisson: five-point stencil

Consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1) \times (0,1), \\ u|_{\partial\Omega} = 0, \end{cases} \qquad -\Delta u = -\partial_{xx} u - \partial_{yy} u.$$

6.8.1 Five-point method

Central differences give

$$-\partial_{xx} u(x,y) \approx \frac{-u(x - \Delta x, y) + 2u(x,y) - u(x + \Delta x, y)}{(\Delta x)^2},$$
$$-\partial_{yy} u(x,y) \approx \frac{-u(x, y - \Delta y) + 2u(x,y) - u(x, y + \Delta y)}{(\Delta y)^2}.$$

At grid point (i,j), with $u_{ij} \approx u(x_i, y_j)$,

$$-\Delta u_{ij} \approx \frac{-u_{i-1,j} + 2u_{ij} - u_{i+1,j}}{(\Delta x)^2} + \frac{-u_{i,j-1} + 2u_{ij} - u_{i,j+1}}{(\Delta y)^2}.$$

Homogeneous Dirichlet: $u_{0,j} = u_{n+1,j} = 0$ and $u_{i,0} = u_{i,m+1} = 0$.

6.8.2 Matrix form

Lexicographic ordering $\underline{u} = (u_{11}, u_{21}, \dots, u_{n1}, u_{12}, \dots, u_{nm})^T$ produces a block-tridiagonal system $A\underline{u} = \underline{f}$. For $\Delta x = \Delta y = h$ and $n = m$, the five-point stencil reads

$$\frac{-u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} + 4u_{ij}}{h^2} = f_{ij}.$$

In block form,

$$A = \begin{pmatrix} A_x + 2I_y & -I_y & 0 & \cdots \\ -I_y & A_x + 2I_y & -I_y & \ddots \\ 0 & -I_y & A_x + 2I_y & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_x = \frac{1}{(\Delta x)^2}\begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & \ddots & \ddots \end{pmatrix}, \qquad I_y = \frac{1}{(\Delta y)^2} I.$$

The discrete energy identity

$$V^T A V = \sum_{i,j}\left[\frac{1}{(\Delta x)^2}(V_{ij} - V_{i-1,j})^2 + \frac{1}{(\Delta y)^2}(V_{ij} - V_{i,j-1})^2\right] \geq 0$$

with equality iff $V = 0$ shows that A is SPD, hence the discrete solution exists uniquely. Truncation error for smooth u is

$$\tau_{ij} = -\frac{(\Delta x)^2}{12}\partial_x^4 u(\xi_i, y_j) - \frac{(\Delta y)^2}{12}\partial_y^4 u(x_i, \eta_j) = O\big((\Delta x)^2, (\Delta y)^2\big).$$

6.8.3 Jacobi iteration

Assuming $\Delta x = \Delta y = h$, rewrite the stencil as

$$\frac{4u_{ij}^{(k+1)}}{h^2} = f_{ij} + \frac{1}{h^2}\big(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\big).$$

In vector form $U^{(k+1)} = \frac{h^2}{4}F + \frac{1}{4}CU^{(k)}$. The exact discrete solution satisfies the same fixed-point relation, so

$$\|U^{(k+1)} - U\| \leq \tfrac{1}{4}\|C\|\,\|U^{(k)} - U\|.$$

In the ∞-norm one has $\|C\|_\infty = 4$, so $\frac{1}{4}\|C\|_\infty = 1$ is not a strict contraction. Convergence of Jacobi follows from the spectral radius $\rho(C/4) = \cos(\pi h) < 1$.

Remark 6.12 (Residual loop)**.** Input $u^{(0)}$; update

$$u_{ij}^{(k+1)} = \frac{h^2}{4}f_{ij} + \frac{1}{4}\big(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\big);$$

set residual $\mathrm{res}^{(k+1)} = \|u^{(k+1)} - u^{(k)}\|$ and iterate until the residual is small.

6.9 Worked examples

Example 6.13 (Exact discrete solution for a sine right-hand side)**.** Consider

$$\begin{cases} -u'' = \sin(\pi x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

The exact solution is $u(x) = \pi^{-2}\sin(\pi x)$. On the uniform grid $x_j = jh$, $h = 1/N$, the load vector is $f_j = \sin(\pi jh)$. The eigenvectors of B are $v_j^{(k)} = \sin(k\pi jh)$ with eigenvalues

$$\lambda_k = \frac{4}{h^2}\sin^2\Big(\frac{k\pi h}{2}\Big), \qquad k = 1, \dots, N-1.$$

Hence the discrete solution is exactly

$$u_j = \frac{\sin(\pi jh)}{\lambda_1} = \frac{h^2\sin(\pi jh)}{4\sin^2(\pi h/2)}.$$

The grid error is therefore a pure amplitude error:

$$\max_j \big|u(x_j) - u_j\big| = \Big|\pi^{-2} - \lambda_1^{-1}\Big|.$$

A short computation gives the second-order table

N	h	$\max_j \lvert u - u_h\rvert$	error$/h^2$
4	1/4	5.37×10^{-3}	8.60×10^{-2}
8	1/8	1.31×10^{-3}	8.40×10^{-2}
16	1/16	3.26×10^{-4}	8.35×10^{-2}
32	1/32	8.14×10^{-5}	8.34×10^{-2}

The ratio error$/h^2$ approaches $\pi^2/120$, consistent with the truncation $\tau = -\frac{h^2}{12}u^{(4)}$ and $\|B^{-1}\|_\infty \le \frac{1}{8}$.

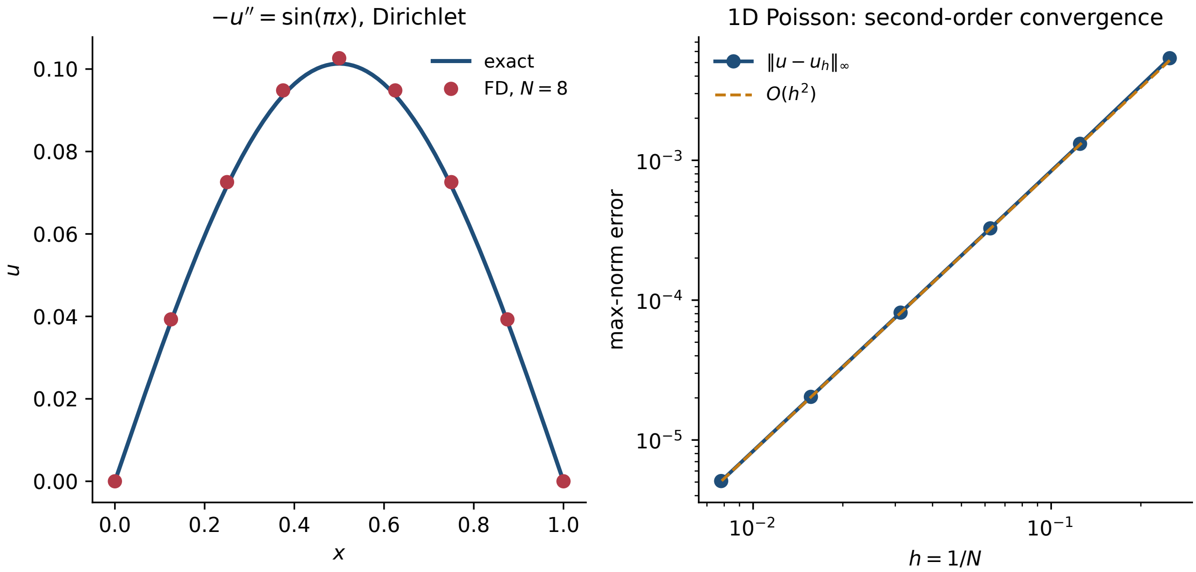


Figure 1: 1D Dirichlet Poisson $-u'' = \sin(\pi x)$. Left: exact solution and the central-difference profile at $N = 8$. Right: max-norm error versus h, with slope two.

Example 6.14 (A 3×3 five-point system)**.** On the unit square take $h = 1/2$, so there is a single interior node $(1/2, 1/2)$. The five-point stencil with homogeneous Dirichlet data reduces to

$$\frac{4u_{11}}{h^2} = f_{11} \qquad\Rightarrow\qquad u_{11} = \frac{h^2}{4}f\big(\tfrac12, \tfrac12\big).$$

If $f \equiv 1$, then $u_{11} = 1/16$. The exact solution of $-\Delta u = 1$ in $(0,1)^2$ with zero boundary values is not elementary, but the same one-point scheme already illustrates uniqueness of the discrete Dirichlet problem (the 1×1 matrix $4/h^2$ is SPD).

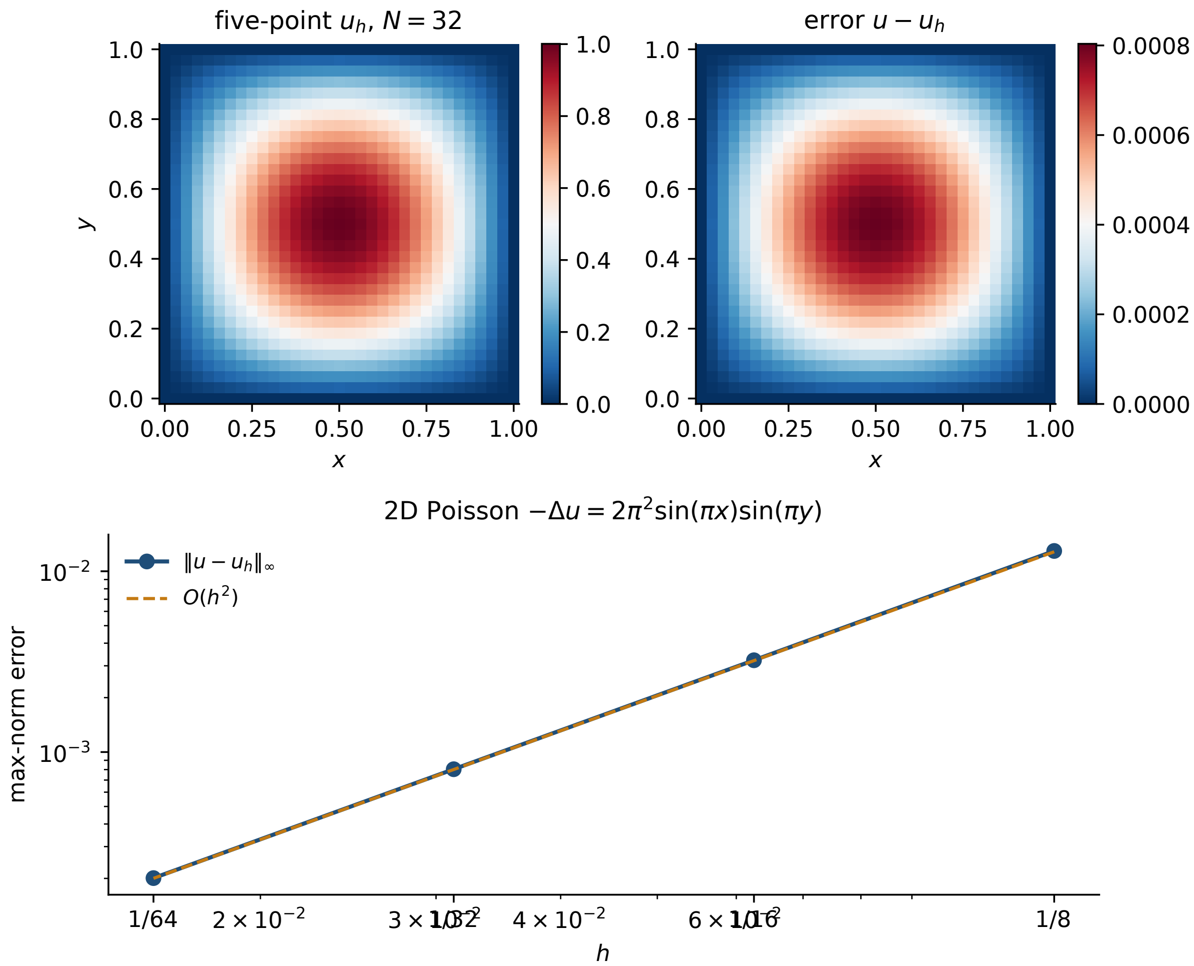


Figure 2: Five-point Poisson for $u = \sin(\pi x)\sin(\pi y)$. Left: discrete solution. Middle: pointwise error. Right: max-norm convergence.

Example 6.15 (Manufactured 2D solution)**.** On $(0,1)^2$ take $u(x,y) = \sin(\pi x)\sin(\pi y)$, so $-\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y)$ with homogeneous Dirichlet data. The five-point matrix is diagonalized by discrete sines; Figure 2 shows the computed field at $N = 32$ and the $O(h^2)$ max-norm error.

Exercise 6.16. For $-u'' = \sin(2\pi x)$ on $(0,1)$ with $u(0) = u(1) = 0$, write down the exact discrete solution on the grid $x_j = jh$ and check that the error is still $O(h^2)$.

Part II
Heat and wave equations

7 The heat equation

Consider the initial-value problem

$$\begin{cases} \partial_t u = \Delta u + f, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = g(x), & x \in \mathbb{R}^n \quad (t=0). \end{cases}$$

7.0.1 Fundamental solution

Assume $f = 0$. By the Fourier transform,

$$\begin{cases} \hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \displaystyle\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, u(x)\, dx, \\ u(x) = (2\pi)^{-\frac{n}{2}} \displaystyle\int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \hat{u}(\xi)\, d\xi. \end{cases}$$

Why $\widehat{\Delta u}(\xi) = -|\xi|^2\hat{u}(\xi)$? First,

$$\begin{aligned} \widehat{\partial_j u}(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \partial_j u\, e^{-ix\cdot\xi}\, dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \Big(\partial_{x_j}\big(u\, e^{-i\xi\cdot x}\big) - u\,(-i\xi_j)\, e^{-ix\cdot\xi}\Big)\, dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} i\xi_j\, u\, e^{-ix\cdot\xi}\, dx \\ &= i\xi_j\, \hat{u}(\xi). \end{aligned}$$

Therefore

$$\widehat{\Delta u}(\xi) = \sum_{j=1}^n \widehat{\partial_{jj} u}(\xi) = \sum_{j=1}^n (i\xi_j)^2\, \hat{u}(\xi) = \sum_{j=1}^n -(\xi_j)^2\, \hat{u}(\xi) = -|\xi|^2\, \hat{u}(\xi).$$

Consequently the heat equation becomes an ODE in t:

$$\begin{cases} \dfrac{d}{dt}\hat{u}(t,\xi) = -|\xi|^2\, \hat{u}(t,\xi), \\ \hat{u}(0,\xi) = \hat{g}(\xi), \end{cases}$$

so

$$\frac{d}{dt}\log\hat{u}(t,\xi) = -|\xi|^2 \qquad \Longrightarrow \qquad \hat{u}(t,\xi) = \hat{g}(\xi)\, e^{-t|\xi|^2}.$$

Equivalently,

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\, \hat{g}(\xi) \qquad \big(\text{or } u(t,x) = e^{t\Delta} g(x)\big).$$

Recall: convolution theorem.

$$\widehat{\varphi * \psi}(\xi) = \hat{\varphi}(\xi)\, \hat{\psi}(\xi)\, (2\pi)^{\frac{n}{2}},$$

where

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\, \psi(y)\, dy.$$

Indeed,

$$\begin{aligned}\widehat{\varphi*\psi}(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \varphi(x-y)\,\psi(y)\,dy\Big) e^{-ix\cdot\xi}\,dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x-y)\,\psi(y)\,e^{-ix\cdot\xi}\,dy\,dx.\end{aligned}$$

By Fubini and the change of variables $z = x - y$,

$$\begin{aligned}\widehat{\varphi*\psi}(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(z)\,\psi(y)\,e^{-iz\cdot\xi}\,e^{-iy\cdot\xi}\,dy\,dz \\ &= (2\pi)^{-\frac{n}{2}} \Big(\int_{\mathbb{R}^n} \varphi(z)\,e^{-iz\cdot\xi}\,dz\Big)\Big(\int_{\mathbb{R}^n} \psi(y)\,e^{-iy\cdot\xi}\,dy\Big) \\ &= \hat{\varphi}(\xi)\,\hat{\psi}(\xi)\,(2\pi)^{\frac{n}{2}}.\end{aligned}$$

Now

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\,\hat{g}(\xi) = (2\pi)^{\frac{n}{2}}\,\widehat{F}(\xi)\,\hat{g}(\xi),$$

so

$$\widehat{F}(\xi) = (2\pi)^{-\frac{n}{2}}\,e^{-t|\xi|^2}.$$

Inverting the Fourier transform,

$$\begin{aligned}\Gamma(x) &= (2\pi)^{-\frac{n}{2}}(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-t|\xi|^2}\,e^{ix\cdot\xi}\,d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-t|\xi|^2+ix\cdot\xi}\,d\xi.\end{aligned}$$

Completing the square,

$$-t|\xi|^2 + ix\cdot\xi = -\Big(\sqrt{t}\,\xi - \frac{ix}{2\sqrt{t}}\Big)^2 - \frac{|x|^2}{4t},$$

hence

$$\Gamma(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-\left(\sqrt{t}\,\xi-\frac{ix}{2\sqrt{t}}\right)^2}\,e^{-\frac{|x|^2}{4t}}\,d\xi.$$

Let

$$\eta = \sqrt{t}\,\xi - \frac{ix}{2\sqrt{t}} \qquad\Longrightarrow\qquad d\xi = \big(t^{-1/2}\big)^n\,d\eta.$$

Then

$$\Gamma(x) = (2\pi)^{-n}\,e^{-\frac{|x|^2}{4t}}\,t^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-|\eta|^2}\,d\eta.$$

The Gaussian factors as a product of one-dimensional integrals:

$$\int_{\mathbb{R}^n} e^{-|\eta|^2}\,d\eta = \Big(\int_{-\infty}^{\infty} e^{-\eta_1^2}\,d\eta_1\Big)\cdots\Big(\int_{-\infty}^{\infty} e^{-\eta_n^2}\,d\eta_n\Big) = (\sqrt{\pi})^n = \pi^{\frac{n}{2}}.$$

Therefore

$$\Gamma(x) = (4\pi t)^{-\frac{n}{2}}\,e^{-\frac{|x|^2}{4t}}.$$

In summary,

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\,\hat{g}(\xi) \qquad\Longrightarrow\qquad u(t,x) = (F_t * g)(x),$$

where

$$F_t(x) = (4\pi t)^{-\frac{n}{2}}\,e^{-\frac{|x|^2}{4t}},$$

so

$$u(t,x) = \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}}\,e^{-\frac{|x-y|^2}{4t}}\,g(y)\,dy.$$

The integrand is the *heat kernel* (Gaussian)

$$K(x,y,t) := (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{|x-y|^2}{4t}},$$

which is singular at $t=0$.

Theorem 7.1. *Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and set*

$$u(t,x) = \int_{\mathbb{R}^n} K(x,y,t)\, g(y)\, dy$$

(the representation exhibits a smoothing effect*). Then:*

(i) $u \in C^\infty$*;*

(ii) $(\partial_t - \Delta_x)u = 0$ *for all* $(t,x) \in (0,\infty)\times\mathbb{R}^n$*;*

(iii)

$$\lim_{(t,x)\to(0,x^0)} u(t,x) = g(x^0) \qquad \forall\, x^0 \in \mathbb{R}^n.$$

Remark 7.2 (Properties of K)**.** The proof follows from basic properties of K:

(a) $K \in C^\infty$;

(b) $(\partial_t - \Delta_x)K(x,y,t) = 0$ for $t > 0$;

(c) $K(x,y,t) > 0$ for $t > 0$;

(d) $\displaystyle\int_{\mathbb{R}^n} K(x,y,t)\, dy = 1$ for all $x \in \mathbb{R}^n$, $t > 0$;

(e) for every $\delta > 0$,

$$\lim_{t\to 0}\int_{|y-x|>\delta} K(x,y,t)\, dy = 0$$

uniformly in $x \in \mathbb{R}^n$.

Proof of Theorem 7.1. (i) & (ii). Since $K \in C^\infty$ with uniformly bounded derivatives of all orders on $[\delta,\infty)\times\mathbb{R}^n$ for each $\delta > 0$, one has $u \in C^\infty$. Moreover

$$\partial_t u - \Delta u = \int_{\mathbb{R}^n} (\partial_t - \Delta_x)K(x,y,t)\, g(y)\, dy = 0,$$

because $(\partial_t - \Delta_x)K = 0$.

(iii). Fix $x^0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Choose $\delta > 0$ small enough that

$$|g(y) - g(x^0)| < \varepsilon \qquad \text{for all } |y - x^0| < \delta.$$

Also, for $|x - x^0| < \delta/2$,

$$u(t,x) - g(x^0) = \int_{\mathbb{R}^n} K(x,y,t)\,\big(g(y) - g(x^0)\big)\, dy,$$

because $\int_{\mathbb{R}^n} K(x,y,t)\, dy = 1$. (One way to see the latter: if

$$\begin{cases} \partial_t u = \Delta u, \\ u(0,x) = 1 \end{cases} \qquad (g \equiv 1),$$

then $u \equiv 1$, since $\partial_t u = 0$.)

Split

$$\begin{aligned}\left|u(t,x)-g(x^0)\right| \le \int_{|y-x^0|<\delta} K(x,y,t)\,|g(y)-g(x^0)|\,dy \\ + \int_{|y-x^0|\ge\delta} K(x,y,t)\,|g(y)-g(x^0)|\,dy =: I_1+I_2.\end{aligned}$$

For the near-field integral,

$$I_1 := \int_{|y-x^0|<\delta} K(x,y,t)\,|g(y)-g(x^0)|\,dy \le \varepsilon \int_{\mathbb{R}^n} K(x,y,t)\,dy = \varepsilon.$$

For the far-field integral: if $|x-x^0|<\delta/2$ and $|y-x^0|\ge\delta$, then

$$|y-x^0| \le |y-x| + |x-x^0| < |y-x| + \tfrac{\delta}{2},$$

and hence

$$\delta \le |y-x^0| \le 2|y-x| \qquad \big(\text{in particular } |y-x| \ge \tfrac{\delta}{2}\big).$$

Therefore

$$\begin{aligned} I_2 &\le \int_{|y-x|\ge\delta/2} 2\max|g|\,(4\pi t)^{-\frac{n}{2}}\, e^{-\frac{|x-y|^2}{4t}}\,dy \\ &\le C\,t^{-\frac{n}{2}} \int_{|y-x|\ge\delta/2} e^{-\frac{|x-y|^2}{4t}}\,dy. \end{aligned}$$

Let $\frac{y-x}{\sqrt{4t}} = z$, so $dy = (\sqrt{4t})^n\,dz$. Then

$$I_2 \lesssim C \int_{|z|\ge\frac{\delta}{4\sqrt{t}}} e^{-|z|^2}\,dz \xrightarrow{t\to 0^+} 0$$

(exponentially fast).

Combining the estimates for I_1 and I_2 yields (iii). □

8 Inhomogeneous heat equation: Duhamel, maximum principle, and uniqueness

8.1 Inhomogeneous heat equation

Consider

$$\begin{cases}\partial_t u - \Delta u = f(t,x) & t>0,\ x\in\mathbb{R}^n,\\ u(0,\cdot) = g(x) & x\in\mathbb{R}^n.\end{cases} \tag{$*$}$$

The candidate solution is

$$u = \int_{\mathbb{R}^n} K(x,y,t)\,g(y)\,dy + \int_0^t\int_{\mathbb{R}^n} K(x,y,t-s)\,f(s,y)\,dy\,ds.$$

Aim: u solves ($*$) and

$$\lim_{(t,x)\to(0,x^0)} u(t,x) = g(x^0).$$

By Duhamel's principle, set

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} K(x,y,t)\,g(y)\,dy,\\ u_2(t,x) &= \int_0^t\int_{\mathbb{R}^n} K(x,y,t-s)\,f(s,y)\,dy\,ds.\end{aligned}$$

By previous arguments,

$$(\partial_t - \Delta_x)u_1 = 0 \qquad \text{and} \qquad \lim_{(t,x)\to(0,x^0)} u_1(t,x) = g(x^0).$$

It suffices to show the following.

Theorem 8.1. *Assume $f \in C_c^\infty\big([0,\infty)\times\mathbb{R}^n\big)$ (with compact support). Define*

$$u_2(t,x) = \int_0^t \int_{\mathbb{R}^n} K(x,y,t-s)\, f(s,y)\, dy\, ds.$$

Then:

(i) $u_2 \in C^0\big([0,\infty)\times\mathbb{R}^n\big)$*;*

(ii) $(\partial_t - \Delta_x)u_2 = f(t,x)$ *for all* $(t,x)\in(0,\infty)\times\mathbb{R}^n$*;*

(iii) $\lim\limits_{(t,x)\to(0,x^0)} u_2(t,x) = 0$ *for all* $x^0\in\mathbb{R}^n$*.*

Proof. (i). Note that $K(x-y,t)$ is singular at $t=0$; we move derivatives onto f to avoid this. That is,

$$\begin{aligned} u_2(t,x) &= \int_0^t \int_{\mathbb{R}^n} (4\pi(t-s))^{-\frac{n}{2}}\, e^{-\frac{|x-y|^2}{4(t-s)}}\, f(s,y)\, dy\, ds \\ &= \tilde{K} * f \qquad \text{in } \mathbb{R}^+\times\mathbb{R}^n \quad (space-timeconvolution). \end{aligned}$$

Change of variables $(s,y)\mapsto(t-s,x-y)$ yields

$$u_2(t,x) = \int_0^t \int_{\mathbb{R}^n} (4\pi s)^{-\frac{n}{2}}\, e^{-\frac{|y|^2}{4s}}\, f(t-s,x-y)\, dy\, ds.$$

Differentiating in t,

$$\begin{aligned} \partial_t u_2(t,x) &= \int_0^t \int_{\mathbb{R}^n} (4\pi s)^{-\frac{n}{2}}\, e^{-\frac{|y|^2}{4s}}\, \partial_t f(t-s,x-y)\, dy\, ds \\ &\quad + \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{|y|^2}{4t}}\, f(0,x-y)\, dy. \end{aligned}$$

Claim.

$$\frac{d}{dt}\int_0^t F(t,s)\, ds = \int_0^t \partial_t F(t,s)\, ds + F(t,t).$$

Indeed, let $s = t\tau$, so $s\in[0,t]$ iff $\tau\in[0,1]$ and $ds = t\, d\tau$. Then

$$\int_0^t F(t,s)\, ds = \int_0^1 F(t,t\tau)\, t\, d\tau,$$

and

$$\begin{aligned} \frac{d}{dt}\int_0^t F(t,s)\, ds &= \int_0^1 \big[t\,\partial_t F(t,t\tau) + t\tau\,\partial_s F(t,t\tau) + F(t,t\tau)\big]\, d\tau \\ &= \int_0^t \partial_t F(t,s)\, ds + \int_0^t \frac{s}{t}\,\partial_s F(t,s)\, ds + \int_0^t F(t,s)\,\frac{1}{t}\, ds. \end{aligned}$$

Integrating by parts,

$$\int_0^t \frac{s}{t}\,\partial_s F(t,s)\, ds = \Big[F(t,s)\,\frac{s}{t}\Big]_{s=0}^t - \int_0^t \frac{1}{t}\, F(t,s)\, ds = F(t,t) - \int_0^t \frac{1}{t}\, F(t,s)\, ds.$$

Hence

$$\frac{d}{dt}\int_0^t F(t,s)\,ds = \int_0^t \partial_t F(t,s)\,ds + F(t,t).$$

In our case

$$F(t,s) = \int_{\mathbb{R}^n} K(y,s)\,f(t-s,x-y)\,dy,$$

so

$$\partial_t u_2(t,x) = \int_0^t \int_{\mathbb{R}^n} K(y,s)\,\partial_t f(t-s,x-y)\,dy\,ds + \int_{\mathbb{R}^n} K(y,t)\,f(0,x-y)\,dy.$$

Similarly,

$$\partial_{x_i} u_2(t,x) = \int_0^t \int_{\mathbb{R}^n} K(y,s)\,\partial_{x_i} f(t-s,x-y)\,dy\,ds.$$

Repeating the process yields

$$u_2 \in C^\infty\big((0,\infty);\, C^\infty(\mathbb{R}^n)\big).$$

(ii). We have

$$\begin{aligned}(\partial_t - \Delta_x)u_2 &= \int_0^t \int_{\mathbb{R}^n} K(y,s)\,(\partial_t f - \Delta_x f)(t-s,x-y)\,dy\,ds \\ &\quad + \int_{\mathbb{R}^n} K(y,t)\,f(0,x-y)\,dy.\end{aligned}$$

Idea: split $(0,t) = (0,\varepsilon)\cup(\varepsilon,t)$, using smallness on $(0,\varepsilon)$ and differentiability on (ε,t). Write

$$(\partial_t - \Delta_x)u_2 = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}I_1 &= \int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s)\,(\partial_t f - \Delta_x f)(t-s,x-y)\,dy\,ds,\\ I_2 &= \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s)\,(\partial_t f - \Delta_x f)(t-s,x-y)\,dy\,ds,\\ I_3 &= \int_{\mathbb{R}^n} K(y,t)\,f(0,x-y)\,dy.\end{aligned}$$

For I_1,

$$|I_1| \le \max\big(|\partial_t f| + |\Delta_x f|\big)\int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s)\,dy\,ds.$$

Since $\int_{\mathbb{R}^n} K(y,s)\,dy = 1$,

$$|I_1| < C\cdot\varepsilon \xrightarrow{\varepsilon\to 0} 0.$$

For I_2,

$$\begin{aligned} I_2 &= \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s)\,(\partial_t f - \Delta_x f)(t-s,x-y)\,dy\,ds \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s)\,(-\partial_s f - \Delta_y f)(t-s,x-y)\,dy\,ds \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \big(-\Delta_y K(y,s)\big)\, f(t-s,x-y)\,dy\,ds \\ &\qquad + \int_\varepsilon^t \int_{\mathbb{R}^n} \partial_s K(y,s)\, f(t-s,x-y)\,dy\,ds \\ &\qquad - \Big[\int_{\mathbb{R}^n} K(y,s)\, f(t-s,x-y)\,dy\Big]_{s=\varepsilon}^t \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} (\partial_s - \Delta_y) K(y,s)\, f(t-s,x-y)\,dy\,ds \\ &\qquad + \int_{\mathbb{R}^n} K(y,\varepsilon)\, f(t-\varepsilon,x-y)\,dy - \int_{\mathbb{R}^n} K(y,t)\, f(0,x-y)\,dy. \end{aligned}$$

For $s>0$ one has $(\partial_s - \Delta_y)K(y,s) = 0$, and the last term equals I_3. Thus

$$I_2 + I_3 = \int_{\mathbb{R}^n} K(y,\varepsilon)\, f(t-\varepsilon,x-y)\,dy = \int_{\mathbb{R}^n} K(x,y,\varepsilon)\, f(t-\varepsilon,y)\,dy.$$

Therefore

$$\begin{aligned} \lim_{\varepsilon\to 0}\big(I_1 + I_2 + I_3\big) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} K(x,y,\varepsilon)\, f(t-\varepsilon,y)\,dy \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} K(x,y,\varepsilon)\, f(t,y)\,dy = f(t,x), \end{aligned}$$

by Theorem 7.1 (recall: $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} K(x,y,\varepsilon)\, f(y)\,dy = f(x)$).

(iii).

$$|u_2(t,x)| \le \|f\|_{L_t^\infty L_x^\infty} \int_0^t \int_{\mathbb{R}^n} K(y,s)\,dy\,ds \le \|f\|_{L_t^\infty L_x^\infty} \int_0^t ds = \|f\|_{L_t^\infty L_x^\infty} \cdot t \xrightarrow{t\to 0} 0.$$

□

Thus

$$u(t,x) = \int_{\mathbb{R}^n} K(x,y,t)\, g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} K(x,y,t-s)\, f(s,y)\,dy\,ds$$

solves the inhomogeneous problem (*Duhamel's principle*).

Remark 8.2 (Differentiating under a moving domain)**.** Let $\Omega(t) \subseteq \mathbb{R}^n$ and let $v(t,x)$ be a velocity field. Then

$$\frac{d}{dt} \int_{\Omega(t)} f\,dx = \int_{\partial\Omega(t)} f\, v\cdot n\, d\sigma_x + \int_{\Omega(t)} \partial_t f\,dx.$$

By Green's formula this equals

$$\int_{\Omega(t)} \big(\operatorname{div}(fv) + \partial_t f\big)\,dx.$$

In fluid mechanics, $\partial_t f + \operatorname{div}(fv)$ is the material derivative Df (material derivative).

Recall the *transport equation*

$$\partial_t f + v\cdot\nabla_x f = 0,$$

which is equivalent to

$$\partial_t f + \operatorname{div}(fv) = 0$$

if $\operatorname{div} v = 0$ (incompressibility), since $\operatorname{div}(fv) = v\cdot\nabla f + f\,\operatorname{div} v$.

8.2 Maximum principle

Let

$$\begin{aligned}\Omega_T &= \{(t,x) \mid x \in \Omega,\ t \in (0,T)\},\\ \partial'\Omega_T &= \{(t,x) \mid x \in \partial\Omega,\ t \in [0,T] \text{ or } x \in \Omega,\ t = 0\} \quad \text{(parabolic boundary)},\\ \partial''\Omega_T &= \{(t,x) \mid x \in \Omega,\ t = T\}.\end{aligned}$$

Theorem 8.3. *If $\partial_t u - \Delta u = f(t,x) \le 0$, then*

$$\max_{\overline{\Omega}_T} u = \max_{\partial'\Omega_T} u.$$

Proof. Step 1. Assume $f < 0$. Let the maximum of u on $\overline{\Omega}_T$ be attained at (t_0, x_0).

Case 1. If $t_0 \in (0,T)$ and $x_0 \in \Omega$, then

$$\begin{cases}\partial_t u(t_0,x_0) = 0,\\ \Delta u(t_0,x_0) \le 0,\end{cases}$$

so

$$(\partial_t u - \Delta u)(t_0,x_0) \ge 0,$$

contradicting $f < 0$.

Case 2. If $t_0 = T$ and $x_0 \in \Omega$, then

$$\partial_t u(T,x_0) \ge 0, \qquad \Delta u(T,x_0) \le 0,$$

so again

$$(\partial_t u - \Delta u)(T,x_0) \ge 0,$$

contradicting $f < 0$.

Hence $t_0 = 0$ with $x_0 \in \Omega$, or $x_0 \in \partial\Omega$, i.e. $(t_0,x_0) \in \partial'\Omega_T$.

Step 2. Now assume $f \le 0$. Set

$$u_\varepsilon(t,x) = u(t,x) + \varepsilon|x|^2.$$

Then

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \partial_t u - \Delta u - 2n\varepsilon = f - 2n\varepsilon < 0.$$

By Step 1,

$$\max_{\overline{\Omega}_T} u_\varepsilon = \max_{\partial'\Omega_T} u_\varepsilon.$$

Let $M := \max_{\partial'\Omega_T} u$ and assume $\Omega \subset B_R(0)$. Then

$$\max_{\overline{\Omega}_T} u_\varepsilon = \max_{\partial'\Omega_T} u_\varepsilon \le M + \varepsilon R^2,$$

so

$$\max_{\overline{\Omega}_T} u \le \max_{\overline{\Omega}_T} u_\varepsilon \le M + \varepsilon R^2 \qquad \forall\, \varepsilon > 0.$$

Letting $\varepsilon \to 0$ yields

$$\max_{\overline{\Omega}_T} u \le M,$$

i.e.

$$\max_{\overline{\Omega}_T} u = \max_{\partial'\Omega_T} u.$$

□

8.3 Uniqueness

There is at most one solution to

$$\begin{cases} \partial_t u - \Delta u = f(t,x) \quad x \in \Omega,\ t \in (0,\infty), \\ u(0,x) = g(x), \\ u\big|_{\partial\Omega} = \varphi(t,x). \end{cases}$$

Proof via the maximum principle. Assume there exist two solutions u_1, u_2, and set $w = u_1 - u_2$. Then w solves

$$\begin{cases} \partial_t w - \Delta w = 0, \\ w(0,x) = 0, \\ w\big|_{\partial\Omega} = 0. \end{cases}$$

By the maximum principle,

$$\max_{\overline{\Omega}_T} w = 0.$$

Considering $-w$ likewise yields $\max_{\overline{\Omega}_T}(-w) = 0$, hence $w \equiv 0$ in Ω_T. □

Uniqueness from energy. Again let $w = u_1 - u_2$, so $w|_{\partial\Omega} = 0$. From $\partial_t w - \Delta w = 0$, integrate $(\partial_t w - \Delta w)w$ over Ω:

$$\begin{aligned} 0 &= \int_\Omega (\partial_t w - \Delta w)\, w\, dx = \int_\Omega \partial_t w \cdot w - (\Delta w)\, w\, dx \\ &= \int_\Omega \tfrac{1}{2}\partial_t |w|^2 + |\nabla w|^2\, dx - \int_{\partial\Omega} w\, \frac{\partial w}{\partial n}\, d\sigma_x \\ &= \tfrac{1}{2}\frac{d}{dt}\int_\Omega |w|^2\, dx + \int_\Omega |\nabla w|^2\, dx, \end{aligned}$$

where the boundary integral vanishes since $w = 0$ on $\partial\Omega$. Hence

$$\frac{d}{dt}\int_\Omega |w|^2(t,x)\, dx = -2\int_\Omega |\nabla w|^2\, dx \le 0.$$

Since $w(0,x) = 0$, one has $\int_\Omega |w|^2(0,x)\, dx = 0$, and therefore

$$0 \le \int_\Omega |w|^2(t,x)\, dx \le \int_\Omega |w|^2(0,x)\, dx = 0.$$

Thus $w \equiv 0$ for all $t > 0$.

Energy and backwards uniqueness. Define

$$E\big(w(t)\big) = \int_\Omega |w|^2(t,x)\, dx.$$

Then

$$\frac{d}{dt}E\big(w(t)\big) = -2\int_\Omega |\nabla w|^2(t,x)\, dx \le 0.$$

Theorem 8.4 (Backwards uniqueness)**.** *Suppose $u_1, u_2 \in C^2(\overline{\Omega}_T)$ solve the heat equation in Ω_T, agree on the lateral boundary $\partial\Omega \times [0,T]$, and $u_1(\,\cdot\,,T) = u_2(\,\cdot\,,T)$. Then $u_1 \equiv u_2$ in Ω_T.*

Proof. Set $w = u_1 - u_2$. Then

$$\frac{d}{dt}E\big(w(t)\big) = -2\int_\Omega |\nabla w|^2\, dx,$$

and

$$\frac{d^2}{dt^2}E\big(w(t)\big) = -2\frac{d}{dt}\int_\Omega |\nabla w|^2\,dx = -4\int_\Omega \nabla w\cdot\nabla w_t\,dx$$
$$= -4\int_\Omega \nabla w\cdot\nabla(\Delta w)\,dx = 4\int_\Omega (\Delta w)^2\,dx,$$

where we used $w_t = \Delta w$.

On the other hand,

$$-2\int_\Omega |\nabla w|^2\,dx = 2\int_\Omega w\,\Delta w\,dx \le 2\Big(\int_\Omega w^2\,dx\Big)^{1/2}\Big(\int_\Omega (\Delta w)^2\,dx\Big)^{1/2}$$

by Hölder's inequality. This shows

$$\big(E'(t)\big)^2 \le E(t)\,E''(t).$$

Let $f(t) = \log E(t)$. Then

$$f'(t) = \frac{E'(t)}{E(t)}, \qquad f''(t) = \frac{E''(t)E(t) - \big(E'(t)\big)^2}{\big(E(t)\big)^2} \ge 0,$$

so f is convex. For $t\in[t_1,t_2]$ with $t = (1-\lambda)t_1 + \lambda t_2$,

$$f\big((1-\lambda)t_1 + \lambda t_2\big) \le (1-\lambda)f(t_1) + \lambda f(t_2),$$

hence

$$E(t) = E\big((1-\lambda)t_1 + \lambda t_2\big) \le E(t_1)^{1-\lambda}\,E(t_2)^{\lambda}.$$

If $t_2 = T$ and $E(T) = 0$, then $E(t)\equiv 0$ for all $0\le t\le T$. □

9 The wave equation

9.1 One-dimensional homogeneous case

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = 0, \\ (u,\partial_t u)(0,x) = (u_0,u_1). \end{cases} \tag{17}$$

Recall by d'Alembert's method:

$$u(t,x) = \frac{1}{2}\big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy. \tag{+}$$

Theorem 9.1. *Assume $(u_0,u_1)\in C^2(\mathbb{R})\times C^1(\mathbb{R})$ and u is defined by (+). Then*

(i) $u\in C^2(\mathbb{R}^+\times\mathbb{R})$;

(ii) $\partial_{tt}u - \partial_{xx}u = 0$ *in* $\mathbb{R}^+\times\mathbb{R}$;

(iii) $\displaystyle\lim_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0)$;

(iv) $\displaystyle\lim_{(t,x)\to(0,x_0)} \partial_t u(t,x) = u_1(x_0)$.

Proof. (i) Differentiating (+),

$$\partial_t u = \frac{1}{2}\big(u_0'(x+t) - u_0'(x-t)\big) + \frac{1}{2}\big(u_1(x+t) + u_1(x-t)\big),$$
$$\partial_{tt} u = \frac{1}{2}\big(u_0''(x+t) + u_0''(x-t)\big) + \frac{1}{2}\big(u_1'(x+t) - u_1'(x-t)\big).$$

Similarly,

$$\partial_x u = \frac{1}{2}\big(u_0'(x+t) + u_0'(x-t)\big) + \frac{1}{2}\big(u_1(x+t) - u_1(x-t)\big),$$
$$\partial_{xx} u = \frac{1}{2}\big(u_0''(x+t) + u_0''(x-t)\big) + \frac{1}{2}\big(u_1'(x+t) - u_1'(x-t)\big).$$

Since $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, one has $\partial_{tt}u, \partial_{xx}u \in C(\mathbb{R}^+ \times \mathbb{R})$.

(ii) Comparing the expressions above yields $\partial_{tt}u - \partial_{xx}u = 0$.

(iii)

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = \frac{1}{2}\big(u_0(x_0) + u_0(x_0)\big) + \frac{1}{2}\int_{x_0}^{x_0} u_1(y)\,dy = u_0(x_0).$$

(iv)

$$\lim_{(t,x)\to(0,x_0)} \partial_t u = \lim_{(t,x)\to(0,x_0)} \Big(\tfrac{1}{2}u_0'(x+t) - \tfrac{1}{2}u_0'(x-t) + \tfrac{1}{2}\big(u_1(x+t) + u_1(x-t)\big)\Big) = u_1(x_0).$$

□

9.2 One-dimensional inhomogeneous case

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = f, \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1). \end{cases} \tag{18}$$

Remark 9.2 (Duhamel's principle for ODEs)**.** For $\dfrac{d^2}{dt^2}u + u = f$,

$$u(t) = S'(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)\,ds,$$

where $S(t) = \sin t$.

By Duhamel's principle we claim

$$\begin{aligned} u(t,x) = &\underbrace{\frac{1}{2}\big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy}_{u_{\mathrm{hom}}} \\ &+ \underbrace{\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\,dy\,ds}_{u_{\mathrm{inh}}}. \end{aligned} \tag{$*$}$$

Theorem 9.3. *Assume $(u_0, u_1, f) \in C^2(\mathbb{R}) \times C^1(\mathbb{R}) \times C^1(\mathbb{R}^+ \times \mathbb{R})$. Then u from $(*)$ satisfies*

(i) $u \in C^2(\mathbb{R}^+ \times \mathbb{R})$;

(ii) $\partial_{tt}u - \partial_{xx}u = f$ *in* $\mathbb{R}^+ \times \mathbb{R}$;

(iii) $\lim_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0)$ *and* $\lim_{(t,x)\to(0,x_0)} \partial_t u(t,x) = u_1(x_0)$.

Proof. It suffices to consider

$$V(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\,dy\,ds$$

(the inhomogeneous part u_{inh}). We will show

(i) $V \in C^2(\mathbb{R}^+ \times \mathbb{R})$;

(ii) $\partial_{tt}V - \partial_{xx}V = f$ in $\mathbb{R}^+ \times \mathbb{R}$;

(iii) $\displaystyle\lim_{(t,x)\to(0,x_0)} V(t,x) = \lim_{(t,x)\to(0,x_0)} \partial_t V(t,x) = 0.$

Remark 9.4. $u = u_{\text{hom}} + \underbrace{u_{\text{inh}}}_{:=V}$; the homogeneous part u_{hom} is treated above.

Set

$$U(t,s,x) = \int_{x-(t-s)}^{x+(t-s)} f(s,y)\,dy,$$

so that $V(t,x) = \dfrac{1}{2}\int_0^t U(t,s,x)\,ds$. Then

$$\partial_t V = \frac{1}{2}\frac{d}{dt}\int_0^t U(t,s,x)\,ds = \frac{1}{2}\int_0^t \partial_t U(t,s,x)\,ds + \frac{1}{2}U(t,t,x).$$

Since $U(t,t,x) = 0$,

$$\partial_t V = \frac{1}{2}\int_0^t \partial_t U(t,s,x)\,ds.$$

Differentiating again,

$$\partial_{tt}V = \frac{1}{2}\int_0^t \partial_{tt}U(t,s,x)\,ds + \frac{1}{2}\partial_t U(t,t,x).$$

Note

$$\partial_t U(t,s,x) = \frac{d}{dt}\int_{x-(t-s)}^{x+(t-s)} f(s,y)\,dy = f\big(s,x+(t-s)\big) + f\big(s,x-(t-s)\big).$$

Hence

$$\partial_t U(t,t,x) = 2f(t,x),$$

and

$$\partial_{tt}U(t,s,x) = \partial_x f\big(s,x+(t-s)\big) - \partial_x f\big(s,x-(t-s)\big).$$

Therefore

$$\partial_{tt}V = f(t,x) + \frac{1}{2}\int_0^t \Big(\partial_x f\big(s,x+(t-s)\big) - \partial_x f\big(s,x-(t-s)\big)\Big)\,ds.$$

Also

$$\partial_x V = \frac{1}{2}\int_0^t \Big(f\big(s,x+(t-s)\big) - f\big(s,x-(t-s)\big)\Big)\,ds,$$
$$\partial_{xx}V = \frac{1}{2}\int_0^t \Big(\partial_x f\big(s,x+(t-s)\big) - \partial_x f\big(s,x-(t-s)\big)\Big)\,ds.$$

Consequently

$$\partial_{tt}V - \partial_{xx}V = f(t,x).$$

Moreover $V \in C^2(\mathbb{R}^+ \times \mathbb{R})$ as long as $f \in C^1(\mathbb{R}^+ \times \mathbb{R})$.

For the initial conditions,

$$\lim_{(t,x)\to(0,x_0)} V(t,x) = \lim_{t\to 0}\frac{1}{2}\int_0^t \int_{x_0-(t-s)}^{x_0+(t-s)} f(s,y)\,dy\,ds = 0,$$
$$\lim_{(t,x)\to(0,x_0)} \partial_t V(t,x) = \lim_{t\to 0}\frac{1}{2}\int_0^t \Big(f\big(s,x_0+(t-s)\big) + f\big(s,x_0-(t-s)\big)\Big)\,ds = 0.$$

□

9.3 Three-dimensional wave equation

$$\begin{cases} \partial_{tt}u - \Delta u = 0 & (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) & x \in \mathbb{R}^3. \end{cases} \tag{19}$$

9.3.1 Spherical mean method

Define the spherical mean

$$\overline{U}(t,x,r) := \frac{1}{4\pi r^2} \int_{\partial B_r(x)} u(t,y)\, d\sigma_y,$$

i.e. the average of u on the sphere $\partial B_r(x)$. In particular,

$$u(x) = \lim_{r \to 0^+} \overline{U}.$$

Similarly,

$$\begin{cases} \overline{U}_0(x,r) = \dfrac{1}{4\pi r^2} \displaystyle\int_{\partial B_r(x)} u_0(y)\, d\sigma_y, \\ \overline{U}_1(x,r) = \dfrac{1}{4\pi r^2} \displaystyle\int_{\partial B_r(x)} u_1(y)\, d\sigma_y. \end{cases} \tag{20}$$

We will show that $\overline{U}(t,x,r)$ solves the radial wave equation (Euler–Poisson–Darboux)

$$\begin{cases} \partial_{tt}\overline{U} - \partial_{rr}\overline{U} - \dfrac{2}{r}\partial_r\overline{U} = 0, \quad t, r \in (0,\infty), \\ \big(\overline{U}, \partial_t\overline{U}\big)\big|_{t=0} = \big(\overline{U}_0, \overline{U}_1\big). \end{cases} \tag{$\star$}$$

Proof. Write

$$\overline{U}(t,x,r) = \frac{1}{4\pi r^2} \int_{\partial B_r(x)} u(t,y)\, d\sigma_y = \frac{1}{4\pi} \int_{|z|=1} u(t, x+rz)\, d\sigma_z.$$

Differentiating in r,

$$\begin{aligned} \partial_r\overline{U}(t,x,r) &= \frac{1}{4\pi} \int_{|z|=1} z \cdot \nabla u(t,x+rz)\, d\sigma_z = \frac{1}{4\pi} \int_{|z|=1} \frac{\partial u}{\partial n}(t,x+rz)\, d\sigma_z \\ &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} \frac{\partial u}{\partial n}(t,y)\, d\sigma_y = \frac{1}{4\pi r^2} \int_{B_r(x)} \Delta u(t,y)\, dy \qquad \text{(by Green's formula)} \\ &= \frac{1}{4\pi r^2} \int_{B_r(x)} \partial_{tt}u(t,y)\, dy \qquad \text{(since } \partial_{tt}u - \Delta u = 0\text{)}. \end{aligned}$$

Hence

$$\begin{aligned} 2r\,\partial_r\overline{U} + r^2\partial_{rr}\overline{U} = \partial_r\big(r^2\partial_r\overline{U}\big) &= \partial_r\Big(\frac{1}{4\pi} \int_{B_r(x)} \partial_{tt}u(t,y)\, dy\Big) \\ &= \frac{1}{4\pi} \int_{\partial B_r(x)} \partial_{tt}u(t,y)\, d\sigma_y \qquad \text{(moving-domain differentiation)} \\ &= r^2\partial_{tt}\overline{U}. \end{aligned}$$

Dividing by r^2 yields ($\star$). □

Then define $\tilde{U} = r\overline{U}$ (radiation field). We find

$$\begin{aligned} \partial_{tt}\tilde{U} - \partial_{rr}\tilde{U} &= r\partial_{tt}\overline{U} - \partial_r\big(\overline{U} + r\partial_r\overline{U}\big) \\ &= r\partial_{tt}\overline{U} - r\partial_{rr}\overline{U} - 2\partial_r\overline{U} = 0. \end{aligned}$$

Therefore

$$\begin{cases} \partial_{tt}\tilde{U} - \partial_{rr}\tilde{U} = 0 & \text{(1D wave)},\\ (\tilde{U}, \partial_t\tilde{U})\big|_{t=0} = (\tilde{U}_0, \tilde{U}_1),\\ \tilde{U}\big|_{r=0} = 0. \end{cases} \tag{21}$$

By the 1D d'Alembert formula,

$$\tilde{U} = \frac{1}{2}\big(\tilde{U}_0(r+t) + \tilde{U}_0(r-t)\big) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{U}_1(\rho)\, d\rho.$$

Then

$$\begin{aligned} u(t,x) &= \lim_{r\to 0^+} \overline{U} = \lim_{r\to 0^+} \big(r^{-1}\tilde{U}\big)\\ &= \frac{1}{2}\lim_{r\to 0^+}\Big(r^{-1}\big(\tilde{U}_0(r+t) + \tilde{U}_0(r-t)\big) + r^{-1}\int_{t-r}^{t+r} \tilde{U}_1(\rho)\, d\rho\Big). \end{aligned}$$

Since $\tilde{U}_0(0) = 0$, take the odd extension $\tilde{U}_0(t-r) = -\tilde{U}_0(r-t)$. The limit becomes

$$u(t,x) = \tilde{U}_0'(t) + \tilde{U}_1(t).$$

Now

$$\tilde{U}_0(t) = \frac{1}{4\pi t}\int_{\partial B_t(x)} u_0(y)\, d\sigma_y = \frac{t}{4\pi}\int_{|z|=1} u_0(x+tz)\, d\sigma_z,$$

so

$$\begin{aligned} \tilde{U}_0'(t) &= \frac{1}{4\pi}\int_{|z|=1} u_0(x+tz)\, d\sigma_z + \frac{t}{4\pi}\int_{|z|=1} z\cdot\nabla u_0(x+tz)\, d\sigma_z\\ &= \frac{1}{4\pi t^2}\int_{\partial B_t(x)} \big[u_0(y) + (y-x)\cdot\nabla u_0(y)\big]\, d\sigma_y. \end{aligned}$$

Therefore Kirchhoff's formula reads

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} \big[u_0(y) + (y-x)\cdot\nabla u_0(y) + t\, u_1(y)\big]\, d\sigma_y. \tag{22}$$

Remark 9.5.

$$u(t_0,x_0) = \frac{1}{4\pi t_0^2}\int_{\partial B_{t_0}(x_0)} \big(u_0 + (y-x_0)\cdot\nabla u_0 + t_0 u_1\big)\, d\sigma_y.$$

One needs information on $|y-x_0| = t_0$ (not the solid ball $|y-x_0| \le t_0$): the light cone. The value at (t_0, x_0) depends on data on the sphere of radius t_0 (domain of dependence); conversely, data at a point x_1 influence the forward light cone (region of influence).

10 Method of descent and energy methods for the wave equation

10.1 2D homogeneous wave equation: method of descent

For $x \in \mathbb{R}^2$, set $\bar{x} = (x, x_3) \in \mathbb{R}^3$ and $\bar{u}(t,\bar{x}) = u(t,x)$, so that $\partial_{x_3}\bar{u} = 0$. If

$$\partial_{tt}u - \Delta_{\mathbb{R}^2}u = 0 \qquad \text{for } x \in \mathbb{R}^2,$$

then

$$\partial_{tt}\bar{u} - \Delta_{\mathbb{R}^3}\bar{u} = 0 \qquad \text{for } \bar{x} \in \mathbb{R}^3.$$

Thus $u(t,x) = \bar{u}(t,x,0)$ is given by the fundamental solution in $\mathbb{R}^3$.

Now take $x_3 = 0$. For the ball $B_t(\bar{x}) \subset \mathbb{R}^3$, points on the sphere $|w - \bar{x}| = t$ may be written

$$w = \big(y,\ \pm\sqrt{t^2 - |y-x|^2}\big).$$

Hence

$$u(t,x) = \frac{1}{4\pi t^2}\int_{|w-\bar{x}|=t}\big(\bar{u}_0(w) + (w-\bar{x})\cdot D_{\bar{x}}\bar{u}_0(w) + t\,\bar{u}_1(w)\big)\,d\sigma_w.$$

Let $\gamma(y) = \sqrt{t^2 - |y-x|^2}$. By symmetry in w_3 (using $\partial_{x_3}\bar{u}_0 = 0$),

$$u(t,x) = \frac{1}{2\pi t^2}\int_{\substack{|w-\bar{x}|=t\\ w_3\ge 0}}\big(\bar{u}_0(w) + (w-\bar{x})\cdot D_{\bar{x}}\bar{u}_0(w) + t\,\bar{u}_1(w)\big)\,d\sigma_w.$$

Recall: surface integral. With $w(y) = (y, \gamma(y))$,

$$\iint_S f(w)\,d\sigma_w = \iint_A f\big(w(y)\big)\sqrt{1 + |\nabla\gamma(y)|^2}\,dy.$$

Then

$$\nabla\gamma(y) = \frac{-(y-x)}{\sqrt{t^2 - |y-x|^2}},$$

so

$$1 + |\nabla\gamma(y)|^2 = 1 + \frac{|y-x|^2}{t^2 - |y-x|^2} = \frac{t^2}{t^2 - |y-x|^2}.$$

Moreover

$$S = \big\{|w-\bar{x}| = t,\ w_3 \ge 0\big\} \iff A = \big\{|y-x| \le t\big\}.$$

Thus

$$\int_S\big(\bar{u}_0(w) + t\,\bar{u}_1(w)\big)\,d\sigma_w = \int_{|y-x|\le t}\big(u_0(y) + t\,u_1(y)\big)\frac{t}{\sqrt{t^2 - |x-y|^2}}\,dy,$$

and, noting $\partial_{x_3}\bar{u}_0 = 0$,

$$\int_S (w-\bar{x})\cdot\nabla_{\bar{x}}\bar{u}_0(w)\,d\sigma_w = \int_{|y-x|\le t}(y-x)\cdot\nabla u_0(y)\frac{t}{\sqrt{t^2 - |x-y|^2}}\,dy.$$

Therefore

$$u(t,x) = \frac{1}{2\pi t^2}\int_{|y-x|\le t}\big(u_0 + (y-x)\cdot Du_0 + t\,u_1\big)\frac{t}{\sqrt{t^2 - |x-y|^2}}\,dy.$$

Remark 10.1 (Duhamel)**.** Duhamel's principle yields the inhomogeneous case

$$\begin{cases}\partial_{tt}u - \Delta u = f,\\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1).\end{cases}$$

In 2D one may write

$$u(t,x) = \frac{1}{2\pi t}\int_{|y-x|\le t}\frac{u_0 + (y-x)\cdot Du_0 + t\,u_1}{\sqrt{t^2 - |x-y|^2}}\,dy$$

and, equivalently,

$$\begin{aligned}u(t,x) = {}&\frac{1}{2\pi}\,\partial_t\int_{|y-x|\le t}\frac{u_0(y)}{\sqrt{t^2 - |x-y|^2}}\,dy + \frac{1}{2\pi}\int_{|y-x|\le t}\frac{u_1(y)}{\sqrt{t^2 - |x-y|^2}}\,dy\\ &+ \frac{1}{2\pi}\int_0^t\int_{|y-x|\le t-s}\frac{f(s,y)}{\sqrt{(t-s)^2 - |x-y|^2}}\,dy\,ds.\end{aligned}$$

10.2 Use of the Fourier transform

Consider

$$\begin{cases} \partial_{tt} u - \Delta u = 0, \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1). \end{cases} \tag{23}$$

In Fourier space,

$$\begin{cases} \partial_{tt} \hat{u}(\xi) + |\xi|^2 \hat{u}(\xi) = 0, \\ \big(\hat{u}(\xi), \partial_t \hat{u}(\xi)\big)\big|_{t=0} = \big(\hat{u}_0(\xi), \hat{u}_1(\xi)\big). \end{cases} \tag{24}$$

Hence

$$\hat{u}(\xi) = A \sin(|\xi| t) + B \cos(|\xi| t).$$

The initial conditions give $\hat{u}_0(\xi) = B$ and $\hat{u}_1(\xi) = A|\xi|$, so $A = \hat{u}_1(\xi)/|\xi|$. Thus

$$\hat{u}(\xi) = \frac{\hat{u}_1(\xi)}{|\xi|} \sin(|\xi| t) + \hat{u}_0(\xi) \cos(|\xi| t).$$

Equivalently, in operator form,

$$u(t, x) = \frac{\sin(|\nabla| t)}{|\nabla|}\, u_1 + \cos(|\nabla| t)\, u_0.$$

10.3 Energy; uniqueness

Theorem 10.2 (Uniqueness)**.** *Let v_1, v_2 solve*

$$\begin{cases} \Box u := \partial_{tt} u - \Delta u = f \qquad \textit{in } \mathbb{R}^n, \\ u \textit{ compactly supported}, \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1). \end{cases} \tag{25}$$

Then $v_1 \equiv v_2$.

Proof. The difference $w = v_1 - v_2$ satisfies

$$\begin{cases} \Box w = \partial_{tt} w - \Delta w = 0 \quad \text{in } \mathbb{R}^n, \\ (w, \partial_t w)\big|_{t=0} = (0, 0). \end{cases} \tag{26}$$

Define

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \big(|\partial_t w|^2 + |\nabla w|^2\big)\, dx.$$

Then

$$\begin{aligned} E'(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \big(2\, \partial_t w \cdot \partial_{tt} w + 2\, \nabla w \cdot \nabla \partial_t w\big)\, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \big(2\, \partial_t w\, (\Delta w) - 2\, \Delta w\, \partial_t w\big)\, dx = 0, \end{aligned}$$

using $\partial_{tt} w = \Delta w$ and integration by parts (compact support). Hence $E(t) = E(0) = 0$, so $w \equiv 0$. □

Theorem 10.3 (Backward uniqueness in the domain of dependence)**.** *If $u = \partial_t u = 0$ on $B_{t_0}(x_0) \times \{t = 0\}$, then $u \equiv 0$ inside the backward cone with tip (t_0, x_0) (domain of dependence).*

Proof. For $t \in [0, t_0]$ define

$$E(t) := \frac{1}{2} \int_{B_{t_0 - t}(x_0)} \big(|\partial_t u|^2 + |\nabla u|^2\big)_{(t,x)}\, dx.$$

Then $E(0) = 0$. Differentiating the integral over the shrinking ball,

$$\begin{aligned} E'(t) &= \int_{B_{t_0-t}(x_0)} \big(\partial_{tt}u \cdot \partial_t u + Du \cdot \partial_t Du\big)\, dx \\ &\quad - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} \big(|\partial_t u|^2 + |Du|^2\big)\, d\sigma_x \\ &= \int_{B_{t_0-t}(x_0)} \partial_t u\,(\partial_{tt}u - \Delta u)\, dx + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial n}\,\partial_t u\, d\sigma_x \\ &\quad - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} \big(|\partial_t u|^2 + |Du|^2\big)\, d\sigma_x \\ &= -\frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} \Big(|\partial_t u|^2 + |Du|^2 - 2\,\frac{\partial u}{\partial n}\,\partial_t u\Big)\, d\sigma_x. \end{aligned}$$

Since $|Du|^2 \ge \big|\nabla u \cdot n\big|^2 = \big|\partial u/\partial n\big|^2$,

$$E'(t) \le -\frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} \Big|\partial_t u - \frac{\partial u}{\partial n}\Big|^2\, d\sigma_x \le 0.$$

Hence $E(t) \le E(0) = 0$, so $E(t) = 0$ and $u \equiv 0$ in the cone. □

11 Finite difference methods for heat and wave equations

The heat and wave equations add a time derivative to the same spatial Laplacian already discretized for Poisson. Time stepping then determines stability: explicit schemes for the heat equation require $\Delta t = O((\Delta x)^2)$, while the wave equation is limited by the CFL condition $c\Delta t/\Delta x \le 1$.

11.1 Heat equation: finite-difference time stepping

Model IBVP (1D heat / parabolic):

$$\begin{cases} \partial_t u = \partial_{xx} u, & t>0,\ 0<x<1, \\ u(0,x) = f(x), & \text{(IC)} \\ u(t,0) = u(t,1) = 0. & \text{(Dirichlet BC)} \end{cases}$$

(Continuous classification, heat kernel / Duhamel, continuous maximum principle and energy estimates: see the heat-equation sections above; omitted here.)

11.1.1 Forward Euler (explicit)

Central difference in space and forward difference in time give truncation $O\big((\Delta x)^2, \Delta t\big)$ and the scheme

$$u_i^{k+1} = u_i^k + \frac{\Delta t}{(\Delta x)^2}\big(u_{i+1}^k + u_{i-1}^k - 2u_i^k\big),$$

with $x_i = i\Delta x$, $t_k = k\Delta t$, $u_i^k \approx u(t_k, x_i)$. Dirichlet: $u_0^k \equiv u_N^k \equiv 0$; Neumann uses ghost points as in the Poisson case. IC: $u_i^0 = f(i\Delta x)$.

Theorem 11.1 (von Neumann stability for Forward Euler)**.** *Substitute $u_n^k = \phi_k e^{in\Delta x\,\theta}$. The growth factor satisfies*

$$\phi_{k+1} = \left(1 - \frac{4\Delta t}{(\Delta x)^2}\sin^2\Big(\frac{\Delta x\,\theta}{2}\Big)\right)\phi_k.$$

Requiring $|\phi_{k+1}| \le |\phi_k|$ for all θ yields

$$\frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}, \qquad \textit{i.e.}\quad \Delta t \le \tfrac{1}{2}(\Delta x)^2.$$

For $\partial_t u = c\partial_{xx} u$, the condition becomes $\Delta t \le \frac{1}{2c}(\Delta x)^2$.

Pros: easy to implement. Cons: restrictive CFL-type constraint; regularity of the update is limited (explicit discrete Laplacian maps $H^2 \to L^2$).

11.1.2 Backward Euler (implicit)

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} = \frac{u_{n+1}^{k+1} + u_{n-1}^{k+1} - 2u_n^{k+1}}{(\Delta x)^2}.$$

With $\lambda = \Delta t/(\Delta x)^2$,

$$AU^{k+1} = U^k, \qquad A = \begin{pmatrix} 1+2\lambda & -\lambda & & \\ -\lambda & 1+2\lambda & -\lambda & \\ & \ddots & \ddots & \ddots \\ & & -\lambda & 1+2\lambda \end{pmatrix}.$$

As in the Poisson case, A is invertible and the scheme is **unconditionally stable** (no restriction on λ: large time steps are allowed). Truncation error remains $O\big(\Delta t, (\Delta x)^2\big)$.

Remark 11.2 (Direct vs. iterative Backward Euler)**. Direct:** unconditional stability and good regularity, but large matrices in higher dimensions. **Iterative** (Jacobi-like on the implicit step):

$$u_j^{(m+1)} = u_j^k + \frac{\Delta t}{(\Delta x)^2}\big[u_{j+1}^{(m)} + u_{j-1}^{(m)} - 2u_j^{(m)}\big],$$

with contraction factor $4\Delta t/(\Delta x)^2$; convergence of the inner iteration requires this factor < 1, so stability/convergence force smaller Δt, though no large matrix solve is needed.

11.1.3 θ-scheme and Crank–Nicolson

For $\theta \in [0,1]$,

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} = \theta\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{(\Delta x)^2} + (1-\theta)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{(\Delta x)^2}.$$

Special cases: $\theta = 0$ Forward Euler; $\theta = 1$ Backward Euler; $\theta = \frac{1}{2}$ Crank–Nicolson.

Expanding about $(x_n, t_{k+1/2})$ shows that Crank–Nicolson cancels the leading $O(\Delta t)$ terms, giving truncation $O\big((\Delta t)^2, (\Delta x)^2\big)$. For general θ,

$$\tau = \Big(\tfrac{1}{2} - \theta\Big)\Delta t\,\partial_{xxt}u + O\big((\Delta t)^2, (\Delta x)^2\big).$$

Hence $\tau = O\big(\Delta t, (\Delta x)^2\big)$ if $\theta \neq \frac{1}{2}$, and $O\big((\Delta t)^2, (\Delta x)^2\big)$ if $\theta = \frac{1}{2}$.

Proposition 11.3 (Stability of the θ-scheme)**.** *Let $\lambda = \Delta t/(\Delta x)^2$. The scheme is*

- *conditionally stable if $0 \le \theta < \frac{1}{2}$, requiring $2\lambda \le 1/(1-2\theta)$;*
- *unconditionally stable if $\frac{1}{2} \le \theta \le 1$.*

Direct Crank–Nicolson: second-order in time, unconditional stability, matrix solve each step. Iterative C–N keeps the same truncation order but needs a smaller Δt for inner convergence.

11.1.4 Higher-order time stepping

BDF methods (with discrete Laplacian $\widetilde{\Delta}$):

$$\begin{aligned}
&\text{BDF1:} && \frac{u^{k+1} - u^k}{\Delta t} = \widetilde{\Delta}u^{k+1} && \to O(\Delta t),\\
&\text{BDF2:} && u^{k+2} - \tfrac{4}{3}u^{k+1} + \tfrac{1}{3}u^k = \tfrac{2}{3}\Delta t\,\widetilde{\Delta}u^{k+2} && \to O((\Delta t)^2),\\
&\text{BDF3:} && u^{k+3} - \tfrac{18}{11}u^{k+2} + \tfrac{9}{11}u^{k+1} - \tfrac{2}{11}u^k = \tfrac{6}{11}\Delta t\,\widetilde{\Delta}u^{k+3} && \to O((\Delta t)^3).
\end{aligned}$$

Classical RK4 for $\dot{y} = f(t,y)$ applied with $f = \widetilde{\Delta}$ yields truncation $O((\Delta t)^4)$ using only one past time level, at the cost of weaker regularity requirements and more work per step.

Example 11.4 (Forward Euler on a sine mode)**.** For $\partial_t u = \partial_{xx}u$ with $u(0,x) = \sin(\pi x)$ and Dirichlet data, the exact solution is $u(t,x) = e^{-\pi^2 t}\sin(\pi x)$. The forward Euler scheme preserves the mode $u_j^k = g^k \sin(\pi j h)$, with

$$g = 1 - 4r\sin^2\Big(\frac{\pi h}{2}\Big), \qquad r = \frac{\Delta t}{h^2}.$$

The exact damping per time step is $e^{-\pi^2\Delta t}$.

- If $r = 0.4$ and $h = 1/10$, then $g \approx 0.961 \in (0,1)$: the mode decays monotonically and the scheme is stable (the worst mode requires $r \le 1/2$).

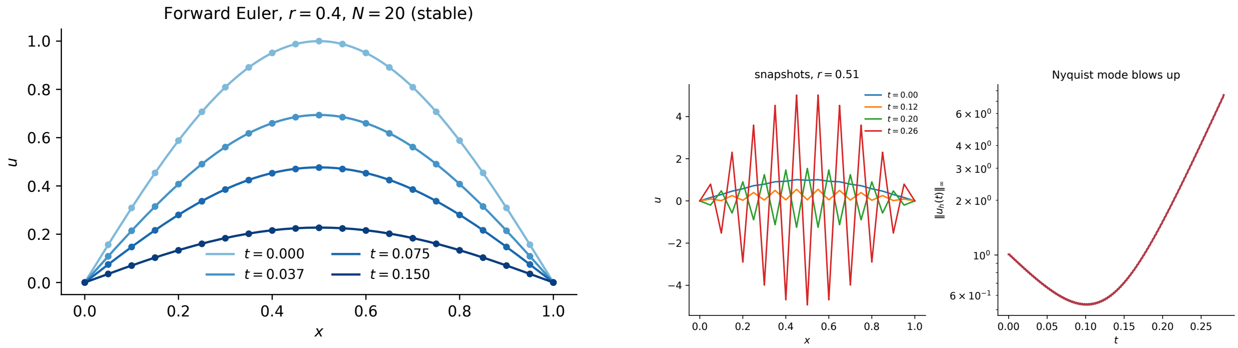


Figure 3: Forward Euler for the heat equation. Left: $r = 0.4$ (markers) against the exact sine mode (curves). Right: $r = 0.51$ with a Nyquist-mode seed; the smooth mode decays at first, then the grid-scale oscillation blows up.

- If $r = 0.51$, the highest grid frequency has amplification $1 - 4r = -1.04$, so $|g| > 1$ and the computation blows up, even though the smooth sine mode itself would still decay.

This is the typical picture: stability is decided by the shortest waves on the mesh, not by the data one intends to resolve.

Example 11.5 (Backward Euler is unconditionally stable)**.** The same sine mode for backward Euler has

$$g = \big(1 + 4r\sin^2(\pi h/2)\big)^{-1} \in (0, 1)$$

for every $r > 0$. With $h = 1/10$ and $r = 5$ (far beyond the explicit CFL), the smooth sine mode still has $g \approx 0.67 \in (0, 1)$, while the highest grid mode has $g = (1 + 4r)^{-1} \approx 0.048$: large time steps are allowed, at the price of strong numerical damping of high frequencies.

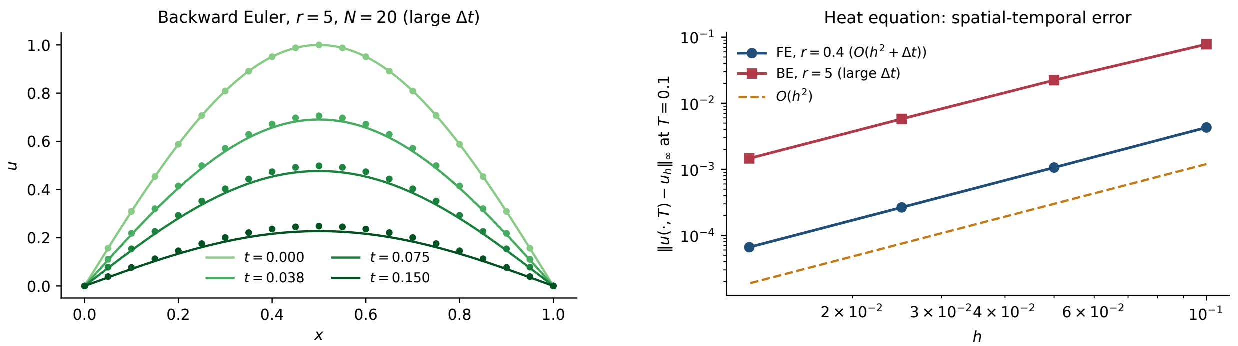


Figure 4: Left: backward Euler with $r = 5$ remains stable, at the cost of extra damping relative to the exact heat kernel. Right: error at $T = 0.1$ versus h; forward Euler with $r = 0.4$ is $O(h^2)$, while a large implicit step is first-order in time.

Exercise 11.6. Implement forward and backward Euler for the example above with $N = 20$ spatial nodes. Compare $u(\,\cdot\,, t = 0.1)$ against $e^{-\pi^2 t}\sin(\pi x)$ for $r = 0.4$ and $r = 0.6$.

11.2 Finite differences for the one-dimensional wave equation

Consider

$$\begin{cases} \partial_{tt} u = c^2 \partial_{xx} u, & t > 0,\ 0 < x < 1, \\ u(t, 0) = u(t, 1) = 0, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x). \end{cases}$$

Centered differences of order two in space and time give the explicit scheme

$$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{(\Delta t)^2}=c^2\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\Delta x)^2},$$

or, with Courant number $\nu=c\Delta t/\Delta x$,

$$u_j^{n+1}=\nu^2u_{j+1}^n+2(1-\nu^2)u_j^n+\nu^2u_{j-1}^n-u_j^{n-1}. \tag{27}$$

The first step uses $u_j^{-1}=u_j^1-2\Delta t\,u_1(x_j)+O((\Delta t)^2)$ together with the scheme at $n=0$, yielding a second-order start

$$u_j^1=u_j^0+\Delta t\,u_1(x_j)+\tfrac{1}{2}\nu^2\big(u_{j+1}^0-2u_j^0+u_{j-1}^0\big).$$

Theorem 11.7 (CFL condition)**.** *A von Neumann analysis of* (27) *produces the amplification roots of* $g^2-2(1-2\nu^2\sin^2(\theta/2))g+1=0$*. One has* $|g|=1$ *for all frequencies if and only if*

$$\nu=\frac{c\Delta t}{\Delta x}\le 1.$$

This is the Courant–Friedrichs–Lewy (CFL) restriction: the numerical domain of dependence must contain the exact light cone.

Example 11.8 (A single sine mode)**.** Take $c=1$, $u_0(x)=\sin(\pi x)$, $u_1\equiv 0$. The exact solution is $u(t,x)=\cos(\pi t)\sin(\pi x)$. On a uniform grid the scheme preserves the spatial mode $u_j^n=a^n\sin(\pi jh)$, and a^n satisfies the recurrence $a^{n+1}-2\big(1-2\nu^2\sin^2(\pi h/2)\big)a^n+a^{n-1}=0$ with $a^0=1$, $a^1=\cos(\omega\Delta t)$ up to $O((\Delta t)^2)$, where $\omega=2h^{-1}\sin(\pi h/2)$ is the discrete frequency. For $\nu=1$ one has $\omega=\pi$ exactly in the $h\to 0$ limit and the scheme reproduces the sampled exact solution of this mode. For $\nu>1$ the recurrence is unstable.

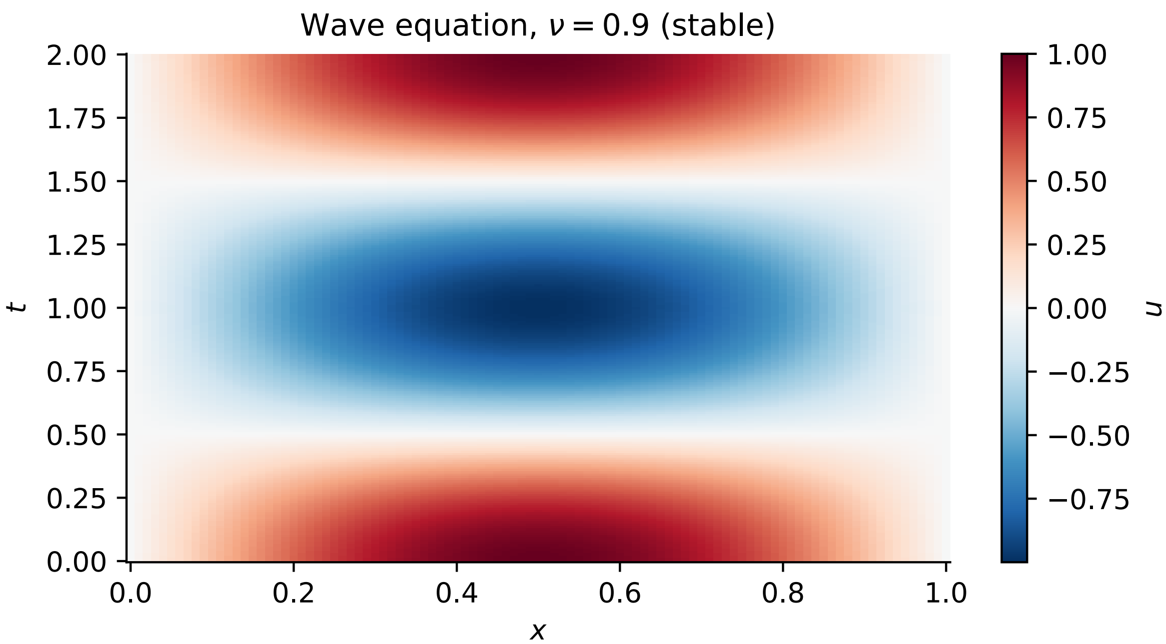


Figure 5: Spacetime plot of the centered wave scheme for $u(0,x)=\sin(\pi x)$, $u_t(0,x)=0$, at Courant number $\nu=0.9$.

Exercise 11.9. Derive the CFL condition $\nu\le 1$ from von Neumann analysis of (27). What happens to a high-frequency mode if $\nu=1.1$?

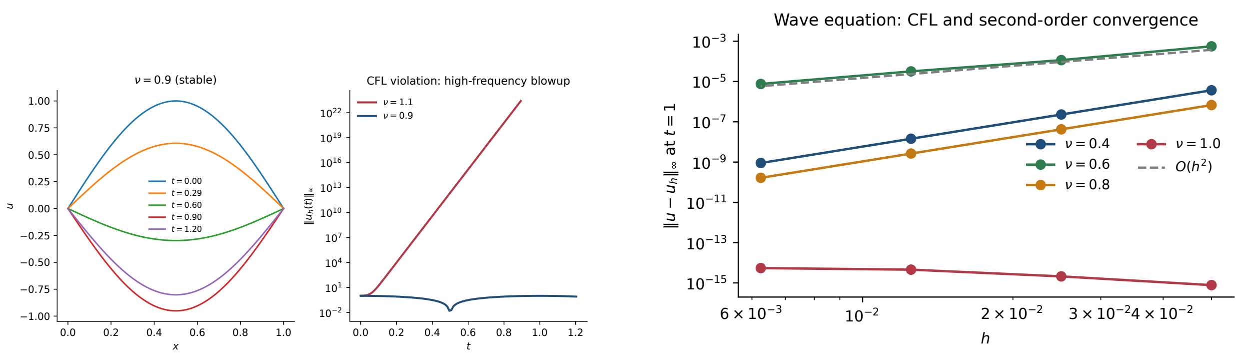


Figure 6: Left: $\nu = 0.9$ remains bounded, while $\nu = 1.1$ with a Nyquist seed grows exponentially. Right: second-order convergence for $\nu < 1$; at $\nu = 1$ the one-dimensional scheme is exact on the grid (machine-roundoff error).

Part III
Sobolev spaces

12 Weak derivatives

12.0.1 Distributions and weak derivatives

Definition 12.1 (L^1_{loc})**.**

$$L^1_{\text{loc}}(\mathbb{R}^n) := \big\{\text{locally integrable functions } f : \mathbb{R}^n \to \mathbb{R}\big\},$$

i.e. $\int_K |f| < +\infty$ for every compact $K \subset \mathbb{R}^n$. The support of a function is

$$\operatorname{supp}(\phi) = \overline{\{x \mid \phi(x) \neq 0\}}.$$

If $f \in C^1$ and $\phi \in C_c^\infty$, then

$$\int_{\mathbb{R}^n} D_i f\, \phi\, dx = -\int_{\mathbb{R}^n} f\, \partial_i \phi\, dx$$

(integration by parts); the right-hand side is well-defined for $\phi \in C_c^\infty$.

Definition 12.2 (Weak derivative)**.** A weak derivative of f is a locally integrable function g such that

$$\int_{\mathbb{R}^n} g(x)\, \phi\, dx = -\int_{\mathbb{R}^n} f(x)\, \partial_i \phi\, dx$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$. We write $g = D_i f$. (Arbitrary changes on a set of measure zero do not affect weak derivatives.)

Exercise 12.3 (Example 1)**.** Let

$$f(x) = \begin{cases} 0 & x \leq 0, \\ x & x > 0. \end{cases}$$

What is the weak derivative of f?

Solution. By definition we need g with

$$\int_{-\infty}^{\infty} f(x)\, \phi'(x)\, dx = -\int_{-\infty}^{\infty} g(x)\, \phi(x)\, dx.$$

Compute

$$\int_{-\infty}^{\infty} f(x)\, \phi'(x)\, dx = \int_0^\infty x\, \phi'(x)\, dx = \big[x\phi(x)\big]_0^\infty - \int_0^\infty \phi(x)\, dx.$$

The boundary term vanishes since $\phi \in C_c^\infty(\mathbb{R})$, so

$$\int_{-\infty}^{\infty} f\, \phi'\, dx = -\int_0^\infty \phi\, dx = -\left(\int_{-\infty}^0 0 \cdot \phi(x)\, dx + \int_0^\infty 1 \cdot \phi(x)\, dx\right).$$

Therefore

$$g(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0 \end{cases}$$

(the Heaviside function). □

Exercise 12.4 (Example 2)**.** What is the weak derivative of the Heaviside function?

It does not exist as an L^1_{loc} function. Indeed, suppose $h \in L^1_{\mathrm{loc}}$ satisfies, for all $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} h\,\phi\,dx = -\int_{-\infty}^{\infty} g\,\phi'\,dx = -\int_0^{\infty} \phi'(x)\,dx = \phi(0).$$

Thus $\int_{-\infty}^{\infty} h\,\phi = \phi(0)$ for all $\phi \in C_c^\infty$. Choose (smooth approximations of)

$$\Phi_n = \begin{cases} 1 & x \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ 0 & \text{otherwise}, \end{cases}$$

so that $\int h\Phi_n = \Phi_n(0) = 1$, but

$$\left|\int_{-\infty}^{\infty} h\Phi_n\,dx\right| \le \int_{-1/n}^{1/n} |h|\,dx.$$

Since $h \in L^1_{\mathrm{loc}}$, the right-hand side tends to 0 as $n \to \infty$, a contradiction.

Remark 12.5. The weak derivative of the Heaviside function is the Dirac delta, which is not an L^1_{loc} function:

- $\delta_0(x) = \begin{cases} \infty & x = 0, \\ 0 & x \neq 0 \end{cases}$ (not a function);
- $\displaystyle\int_{-\infty}^{\infty} \delta_0(x)\,dx = 1;$
- $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta_0(x)\,dx = f(0).$

12.0.2 Properties of weak derivatives in $\mathbb{R}^n$ and $W^{k,p}$ spaces

Theorem 12.6 (Uniqueness of weak derivatives)**.** *If $g_1, g_2 = D^i f$ are two weak derivatives of f, then*

$$\int_{\mathbb{R}^n} g_1\phi\,dx = -\int_{\mathbb{R}^n} f\,\partial^i\phi\,dx = \int_{\mathbb{R}^n} g_2\phi\,dx,$$

so

$$\int_{\mathbb{R}^n} (g_1 - g_2)\phi\,dx = 0 \qquad \forall\,\phi \in C_c^\infty(\mathbb{R}^n),$$

hence $g_1 \equiv g_2$.

Theorem 12.7 (Commutativity)**.** *$D^\alpha(D^\beta f) = D^\beta(D^\alpha f) = D^{\alpha+\beta} f$.*

Proof.

$$\int D^\alpha(D^\beta f)\,\phi\,dx = (-1)^{|\alpha|}\int D^\beta f\,\partial^\alpha\phi\,dx = \cdots = (-1)^{|\alpha|+|\beta|}\int f\,\partial^{\alpha+\beta}\phi\,dx.$$

□

Theorem 12.8 (Stability under L^1_{loc} limits)**.** *If $f_n \to f$ in L^1_{loc} and $D^\alpha f_n \to g$ in L^1_{loc}, then $D^\alpha f = g$.*

Proof. For every $\phi \in C_c^\infty(\mathbb{R}^n)$ we aim to show

$$(-1)^{|\alpha|}\int f\,\partial^\alpha\phi\,dx = \int g\,\phi\,dx.$$

We know

$$(-1)^{|\alpha|}\int f_n\,\partial^\alpha\phi\,dx = \int D^\alpha f_n\,\phi\,dx.$$

Since $f_n \to f$ in L^1_{loc},

$$\lim_n (-1)^{|\alpha|}\int f_n\,\partial^\alpha\phi = (-1)^{|\alpha|}\int f\,\partial^\alpha\phi.$$

Since $D^\alpha f_n \to g$ in L^1_{loc},

$$\lim_n \int D^\alpha f_n\,\phi = \int g\,\phi.$$

□

13 Sobolev spaces $W^{k,p}$: approximation, extension, and embeddings

13.1 Sobolev spaces $W^{k,p}(\mathbb{R}^n)$

13.1.1 L^p spaces $(1 \le p \le \infty)$

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^p\,dx\right)^{1/p}, \qquad \|f\|_{L^\infty(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f|$$

(in the measure-theoretic sense).

Important inequalities.

- Hölder's inequality: if $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_{\mathbb{R}^n} |uv|\,dx \le \left(\int_{\mathbb{R}^n} |u|^p\,dx\right)^{1/p}\left(\int_{\mathbb{R}^n} |v|^q\,dx\right)^{1/q},$$

  i.e. $\|uv\|_{L^1(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)}\|v\|_{L^q(\mathbb{R}^n)}$. On a finite-measure domain one also has $\|u\|_{L^{p_1}(\Omega)} \le C\|u\|_{L^{p_2}(\Omega)}$ (for $p_1 \le p_2$).

- Minkowski inequality:

$$\|u+v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Remark 13.1 (L^1_{loc} vs. L^1)**.**

$$L^1_{\text{loc}}(\mathbb{R}^n): \quad \forall \text{ compact } K\subset\mathbb{R}^n, \quad \int_K |f|\,dx < +\infty,$$
$$L^1(\mathbb{R}^n): \quad \int_{\mathbb{R}^n} |f|\,dx < +\infty.$$

Example: $f(x)=x$ on $\mathbb{R}$ satisfies $f\notin L^1(\mathbb{R})$ since $\int_{-\infty}^{\infty}|x|\,dx=+\infty$, but for every $[a,b]\subset\mathbb{R}$ one has $\int_a^b |x|\,dx<+\infty$, so $f\in L^1_{\text{loc}}(\mathbb{R})$. Indeed $L^1(\mathbb{R}^n)\subset L^1_{\text{loc}}(\mathbb{R}^n)$: for compact $K\subset\mathbb{R}^n$,

$$\int_K |f|\,dx \le \int_{\mathbb{R}^n}|f|\,dx = \|f\|_{L^1(\mathbb{R}^n)} < +\infty.$$

Similarly one can define $L^p(\Omega)$ for $\Omega\subseteq\mathbb{R}^n$.

13.1.2 Definition of $W^{k,p}(\Omega)$

$$W^{k,p}(\Omega) = \big\{ u \bigm| D^\alpha u \in L^p(\Omega)\ \forall\, |\alpha| \le k \big\},$$

with

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)},$$

or equivalently

$$\left(\int_\Omega \sum_{|\alpha|\le k} |D^\alpha u|^p \, dx \right)^{1/p}.$$

Define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{k,p}}},$$

i.e. the closure of C_c^∞ functions under the $W^{k,p}$ norm. Thus for every $f \in W_0^{k,p}(\Omega)$ there exist $f_n \in C_c^\infty(\Omega)$ with $\|f_n - f\|_{W^{k,p}(\Omega)} \to 0$.

Example 13.2. Take $\Omega = (-1,1)$ and $u = |x|$. Then

$$Du = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0, \end{cases} \implies Du \in L^p(\Omega) \quad \forall\, p \ge 1,$$

but $D^2 u$ does not exist in any L^p. Indeed, for every $\phi \in C_c^\infty((-1,1))$,

$$\begin{aligned} \int_{-1}^1 D^2 u \, \phi \, dx &= -\int_{-1}^1 Du\, \phi'(x) \, dx \\ &= -\int_{-1}^0 (-1)\phi'(x)\, dx - \int_0^1 \phi'(x)\, dx \\ &= \int_{-1}^0 \phi'(x)\, dx - \int_0^1 \phi'(x)\, dx \\ &= \big(\phi(0) - \phi(-1)\big) - \big(\phi(1) - \phi(0)\big) = 2\phi(0), \end{aligned}$$

since $\phi(1) = \phi(-1) = 0$. Hence $\int_{-1}^1 D^2 u\, \phi\, dx = 2\phi(0)$, i.e. $D^2 u = 2\delta_0$.

Example 13.3. Let $\Omega = B_1(0) \subseteq \mathbb{R}^n$ and $u(x) = |x|^{-\alpha}$. Then

$$u \in W^{1,p}(\Omega) \iff \alpha < \frac{n-p}{p}.$$

Proof.

(1) $u \in L^p(\Omega)$ iff $\int_{B_1(0)} (|x|^{-\alpha})^p \, dx < +\infty$. In polar coordinates,

$$\int_{B_1(0)} |x|^{-\alpha p} \, dx = \int_0^1 \int_{\partial B_\rho(0)} \rho^{-\alpha p} \, dS \, d\rho = C \int_0^1 \rho^{-\alpha p} \rho^{n-1} \, d\rho = C \big[\rho^{-\alpha p + n}\big]_{\rho=0}^1.$$

As $\rho \to 0$ this is meaningful iff $-\alpha p + n \ge 0$, i.e. $\alpha \le n/p$.

(2) $Du \in L^p(\Omega)$: $Du = (-\alpha)|x|^{-\alpha-1} D|x|$, and $D_i|x| = x_i/|x|$ for $x \ne 0$ (cf. the 1D case $D|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$). Thus $Du = (-\alpha)|x|^{-\alpha-2}x$, and

$$\begin{aligned} Du \in L^p(\Omega) &\iff \int_{B_1(0)} \big| |x|^{-\alpha-2} x \big|^p \, dx < +\infty \iff \int_{B_1(0)} |x|^{-\alpha p - p} \, dx < +\infty \\ &\iff \int_0^1 \rho^{-\alpha p - p + n - 1} \, d\rho < +\infty \iff \big[\rho^{-\alpha p - p + n}\big]_{\rho=0}^1 < +\infty, \end{aligned}$$

i.e. $-\alpha p - p + n \ge 0$, or $\alpha \le (n-p)/p$.

Remark 13.4. $W^{k,p}(\Omega)$ is a Banach space (a complete normed space):

$$\|cu\| = |c|\,\|u\| \quad \forall\, c \in \mathbb{C}, \qquad \|u\| = 0 \iff u = 0, \qquad \|u+v\| \le \|u\| + \|v\|.$$

Special case $p = 2$: $H^k := W^{k,2}$. Completeness means every Cauchy sequence converges; in particular L^p is complete.

13.2 Approximation by smooth functions

Define the standard mollifier

$$\eta(x) = \begin{cases} C\exp\Big(\dfrac{1}{|x|^2-1}\Big) & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

so that $\exp\big(1/(|x|^2-1)\big) \to e^{-\infty} = 0$ as $|x| \to 1$. Choose $C > 0$ so that $\int_{\mathbb{R}^n} \eta\, dx = 1$. Then set

$$\eta^\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big) \qquad \big(\text{height}\uparrow,\ \text{width}\downarrow\big).$$

A change of variables $y = x/\varepsilon$ gives

$$\int_{\mathbb{R}^n} \eta^\varepsilon(x)\, dx = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big)\, dx = \int_{|x|\le\varepsilon} \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big)\, dx = \int_{|y|\le 1} \eta(y)\, dy = 1.$$

We now define $u^\varepsilon = \eta^\varepsilon * u$ (mollifier).

Theorem 13.5. *$u^\varepsilon \in C^\infty(\Omega_\varepsilon)$, where $\Omega_\varepsilon = \{d(x,\partial\Omega) > \varepsilon\}$, and $u^\varepsilon \to u$ in $W^{k,p}_{\text{loc}}(\Omega)$.*

Proof.

$$u^\varepsilon(x) = \eta^\varepsilon * u(x) = \int_{|y-x|\le\varepsilon} \frac{1}{\varepsilon^n}\eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\, dy,$$

and

$$u^\varepsilon(x) - u(x) = \int_{|y-x|\le\varepsilon} \frac{1}{\varepsilon^n}\eta\Big(\frac{y-x}{\varepsilon}\Big)\big(u(y)-u(x)\big)\, dy.$$

If u is continuous, then for every $\delta > 0$ one has $|u(y) - u(x)| < \delta$ whenever $|y - x| < \varepsilon$ (for ε small enough), whence

$$\big|u^\varepsilon(x) - u(x)\big| \le \delta \int_{|y-x|\le\varepsilon} \frac{1}{\varepsilon^n}\eta\Big(\frac{y-x}{\varepsilon}\Big)\, dy = \delta \to 0.$$

Thus $u^\varepsilon(x) \to u(x)$ pointwise as $\varepsilon \to 0$. Also

$$\partial_{x_i} u^\varepsilon(x) = \int_{|y-x|\le\varepsilon} \frac{1}{\varepsilon^n}\cdot\frac{1}{\varepsilon}\partial_{y_i}\eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\, dy$$

(bounded derivatives), so $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$. Similarly $D^\alpha u^\varepsilon \to D^\alpha u$ pointwise; the dominated convergence theorem finishes the argument. □

Theorem 13.6 (Global approximation of $W^{k,p}(\Omega)$ by C^∞)**.** *Assume Ω is bounded, $u \in W^{k,p}(\Omega)$, and $p \in [1,\infty)$. Then there exist $u_n \in C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ such that $\|u_n - u\|_{W^{k,p}(\Omega)} \to 0$.*

(Proof omitted; density theorem.)

Remark 13.7. $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

13.3 Extension and trace

Extension.

Theorem 13.8 (Extension)**.** *Assume Ω is bounded and $\partial\Omega$ is C^1. Select an open bounded set V with $\Omega \subset\subset V$. Then for any $u \in W^{1,p}(\Omega)$ there exists an extension $Eu \in W^{1,p}(\mathbb{R}^n)$ such that*

(i) $Eu = u$ *a.e. in* Ω*;*

(ii) $\operatorname{supp}(Eu) \subset V$*;*

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$.

In particular E is a bounded linear operator.

Idea of the proof. *Step 1.* Fix $x^0 \in \partial\Omega$ and suppose $\partial\Omega$ is flat near x^0 (locally flatten). Let $B = B_r(x^0)$ and

$$B^+ := B \cap \{x_n \ge 0\} \subseteq \overline{\Omega}, \qquad B^- := B \cap \{x_n \le 0\}.$$

The function u is defined in B^+; the aim is to define Eu on B^-. Set

$$\bar{u}(x) := \begin{cases} u(x) & x = (x_1, \dots, x_n) \in B^+, \\ -3u(x', -x_n) + 4u\big(x', -\frac{x_n}{2}\big) & x = (x_1, \dots, x_n) \in B^-, \end{cases}$$

where $x' = (x_1, \dots, x_{n-1})$. Continuity across $\{x_n = 0\}$:

$$\lim_{x_n \to 0^+} \bar{u}(x) = \lim_{x_n \to 0^-} \bar{u}(x),$$

since at $x_n = 0$ one has $-3u(x', 0) + 4u(x', 0) = u(x', 0)$. Moreover $\partial_{x_n}\bar{u}$ is continuous across $\{x_n = 0\}$: on B^-,

$$\partial_{x_n}\Big(-3u(x', -x_n) + 4u\big(x', -\tfrac{x_n}{2}\big)\Big) = 3\partial_{x_n}u(x', -x_n) - 2\partial_{x_n}u\big(x', -\tfrac{x_n}{2}\big),$$

which equals $\partial_{x_n}u$ at $x_n = 0$. It is also not hard to check $\|\bar{u}\|_{W^{1,p}(B)} \le C\|u\|_{W^{1,p}(B^+)}$.

Step 2. $\partial\Omega$ is C^1: locally straighten (differential geometry).

Step 3. $\partial\Omega$ is compact: covers by finitely many balls B.

Trace. Question: given $u \in W^{1,p}(\Omega)$, how do we define $u|_{\partial\Omega}$? Here Ω is n-dimensional while $\partial\Omega$ is $(n-1)$-dimensional (and $\partial\Omega$ has measure zero in $\mathbb{R}^n$).

Theorem 13.9 (Trace)**.** *Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

(1) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$*;*

(2) $\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$.

Idea of the proof. Assume $x^0 \in \partial\Omega$ and $\partial\Omega$ is flat near x^0. Let $B = B_r(x^0)$, $\hat{B} = B_{r/2}(x^0)$, and pick $\zeta \in C_c^\infty(B)$ with

$$\zeta = \begin{cases} 1 & \text{in } \hat{B}, \\ 0 & \text{outside } B. \end{cases}$$

Write $\Gamma := \partial\Omega \cap \hat{B}$ and $x' = (x_1, \dots, x_{n-1})$. Then

$$\begin{aligned}
\int_{\Gamma} |u|^p \, dx' &\le \int_{\{x_n=0\}} \zeta |u|^p \, dx' = \int_{\{x_n=0\}} 1 \cdot \zeta |u|^p \, dx' \qquad \big(functionality\big) \\
&= -\int_{B^+} \partial_{x_n}\big(\zeta |u|^p\big) \, dx \qquad \text{(Green's formula)} \\
&= -\int_{B^+} \Big[|u|^p \partial_{x_n} \zeta + p|u|^{p-1} \operatorname{sgn}(u) \cdot \partial_{x_n} u \cdot \zeta \Big] \, dx \\
&\le C \int_{B^+} |u|^p + |Du|^p \, dx.
\end{aligned}$$

(Young: $ab \le a^\alpha/\alpha + b^\beta/\beta$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, taking $\alpha = p$ and $\beta = p/(p-1)$.)

Then for non-flat $\partial\Omega$:

- straighten $\Rightarrow \displaystyle\int_{\Gamma} |u|^p \, d\sigma \le C \int_{\Omega} |u|^p + |Du|^p \, dx$;
- compactness, finite cover $\Rightarrow \|u\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$.

Theorem 13.10 (Trace-zero functions in $W^{1,p}(\Omega)$)**.** *Assume Ω is bounded, $\partial\Omega$ is C^1, and $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$ on $\partial\Omega$.*

Recall $W_0^{1,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}}}$.

Proof. ($\Rightarrow$) If $u \in W_0^{1,p}(\Omega)$, there exist $u_m \in C_c^\infty(\Omega)$ with $u_m \to u$ in $W^{1,p}$, i.e. $\|u_m - u\|_{W^{1,p}(\Omega)} \to 0$. Then

$$\|Tu_m - Tu\|_{L^p(\partial\Omega)} \le C\|u_m - u\|_{W^{1,p}(\Omega)} \to 0.$$

Since $Tu_m = 0$ (as $u_m \in C_c^\infty(\Omega)$), one gets $Tu = 0$ on $\partial\Omega$.

($\Leftarrow$) If $Tu = 0$ on $\partial\Omega$, aim to construct $u_m \in C_c^\infty(\Omega)$ with $u_m \to u$ in $W^{1,p}$. WLOG (locally flat + extension) we assume

(1) $u \in W^{1,p}(\mathbb{R}^n_+)$ has compact support;

(2) $Tu = 0$ on $\{x_n = 0\}$.

Then apply a mollifier to construct u_m … (omitted). □

13.4 Sobolev embeddings

$X \subset Y$ embedding means $\|f\|_Y \le C\|f\|_X$, so $f \in X$ with $\|f\|_X < +\infty$ implies $f \in Y$ with $\|f\|_Y < +\infty$.

Question: $W^{1,p}(\Omega) \subset$? depends on p:

$$\begin{cases} 1 \le p < n, \\ p = n, \\ n < p \le \infty. \end{cases}$$

13.4.1 Gagliardo–Nirenberg–Sobolev inequality

Case $1 \le p < n$: we want

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \qquad (*)$$

where C and q do not depend on u.

Motivation. If ($*$) holds for every u, it must hold for $u_\lambda(x) = u(\lambda x)$, $x \in \mathbb{R}^n$, i.e.

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}.$$

Now

$$\int_{\mathbb{R}^n} |u_\lambda|^q \, dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx = \int_{\mathbb{R}^n} |u(y)|^q \frac{dy}{\lambda^n},$$

and

$$\int_{\mathbb{R}^n} |Du_\lambda|^p \, dx = \int_{\mathbb{R}^n} \lambda^p |Du(\lambda x)|^p \, dx = \lambda^{p-n} \int_{\mathbb{R}^n} |Du(y)|^p \, dy.$$

Thus ($*$) becomes

$$\lambda^{-n/q}\|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{(p-n)/p}\|Du\|_{L^p(\mathbb{R}^n)},$$

or

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}.$$

This should be independent of λ! So

$$1 - \frac{n}{p} + \frac{n}{q} = 0$$

(otherwise let $\lambda \to \infty$ or $\lambda \to 0$). Hence

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n} \qquad \text{or} \qquad q = \frac{np}{n-p} =: p^*,$$

which requires $1 \le p < n$.

Theorem 13.11 (Gagliardo–Nirenberg–Sobolev)**.** *Assume* $1 \le p < n$. *Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall\, u \in C_c^1(\mathbb{R}^n).$$

Remark 13.12. This version needs u to have compact support! (Counterexample: $u \equiv 1$ gives LHS $= +\infty$, RHS $= 0$.) But C does not depend on the support.

Proof. Step 1. Assume $p = 1$ (so $1^* = n/(n-1)$). By the fundamental theorem of calculus,

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \, dy_i,$$

hence

$$|u(x)| \le \int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)| \, dy_i \qquad (i = 1, \dots, n).$$

So

$$|u(x)|^{n/(n-1)} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)| \, dy_i \right)^{1/(n-1)}. \tag{$*$}$$

Integrate ($*$) with respect to x_1:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^{n/(n-1)} \, dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du| \, dy_i \right)^{1/(n-1)} dx_1 \\
&= \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{1/(n-1)} \int_{-\infty}^{\infty} \prod_{i=2}^n \left(\int_{-\infty}^{\infty} |Du| \, dy_i \right)^{1/(n-1)} dx_1
\end{aligned}$$

(independent of x_1; apply Hölder in x_1 with $p_i = n-1$). Thus

$$\le \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{1/(n-1)} \prod_{i=2}^n \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dy_i \, dx_1 \right)^{1/(n-1)}.$$

Again integrate with respect to x_2 (factor out; $i = 2$ term independent of x_2; apply Hölder in x_2):

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{n/(n-1)}\,dx_1\,dx_2 \\
&\le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dx_1\,dy_2\right)^{1/(n-1)} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dy_1\,dx_2\right)^{1/(n-1)} \\
&\quad \cdot \prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dx_1\,dx_2\,dy_i\right)^{1/(n-1)}.
\end{aligned}$$

Repeating the process,

$$\int_{\mathbb{R}^n} |u|^{n/(n-1)}\,dx \le \left(\prod_{i=1}^{n}\int_{\mathbb{R}^n} |Du|\,dx\right)^{1/(n-1)} = \left(\int_{\mathbb{R}^n} |Du|\,dx\right)^{n/(n-1)},$$

i.e. $\|u\|_{L^{1^*}(\mathbb{R}^n)} \le \|Du\|_{L^1(\mathbb{R}^n)}$.

Step 2. Now for $1 < p < n$, apply the $p = 1$ case to $v = |u|^\gamma$:

$$\begin{aligned}
\|v\|_{L^{n/(n-1)}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |u|^{\gamma n/(n-1)}\,dx\right)^{(n-1)/n}, \\
\|Dv\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \gamma |u|^{\gamma-1}|Du|\,dx \\
&\le \gamma\left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)p/(p-1)}\,dx\right)^{(p-1)/p} \|Du\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Choose $\gamma n/(n-1) = (\gamma-1)p/(p-1)$, i.e. $\gamma = p(n-1)/(n-p) > 1$. Then

$$\frac{\gamma n}{n-1} = \frac{np}{n-p} = p^*,$$

and

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{(n-1)/n-(p-1)/p} \le \|Du\|_{L^p(\mathbb{R}^n)}.$$

Since $(n-1)/n - (p-1)/p = (n-p)/(np) = 1/p^*$, we obtain

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}.$$

□

Corollary 13.13. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded with $\partial\Omega$ of class C^1, and $1 \le p < n$. Then for every $q \in [1, p^*]$ with $p^* = np/(n-p)$,*

$$\|f\|_{L^q(\Omega)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

Proof. Let $\tilde{\Omega} = \{x \in \mathbb{R}^n : d(x,\Omega) < 1\}$ be an open neighborhood of Ω. Apply the extension theorem:

$$\|f\|_{L^q(\Omega)} \le C_1\|f\|_{L^{p^*}(\Omega)} \le C_2\|Ef\|_{L^{p^*}(\mathbb{R}^n)} \le C_3\|f\|_{W^{1,p}(\Omega)}$$

(by Hölder, then GNS). □

13.4.2 Morrey's estimate

Question: what about $p = n$ and $p > n$? Borderline $p = n$: $p^* = np/(n-p) \to +\infty$ as $p \to n$. Does $W^{1,n}(\Omega) \subseteq L^\infty(\Omega)$? **No for $n > 1$.**

Example 13.14. Take $n = 2$ and $u = \log\log\bigl(1+1/|x|\bigr)$. Then $u \notin L^\infty(B_1(0))$ (as $x \to 0$, $u \to \infty$), but $u \in W^{1,2}(B_1(0))$.

1°. $u \in L^2(B_1(0))$:

$$\begin{aligned}\int_{B_1(0)} \bigl[\log\log(1+1/|x|)\bigr]^2\,dx &= \int_0^{2\pi}\int_0^1 \bigl(\log\log(1+1/\rho)\bigr)^2 \rho\,d\rho\,d\theta \\ &\le C\int_0^1 \bigl(\log\log(1+1/\rho)\bigr)^2\rho\,d\rho \qquad \bigl(\textit{bounded}\bigr).\end{aligned}$$

Indeed

$$\lim_{\rho\to 0}\bigl(\log\log(1+1/\rho)\bigr)^2\rho = \lim_{\rho\to 0}\frac{\bigl(\log\log(1+1/\rho)\bigr)^2}{1/\rho} = 0$$

(by L'Hôpital twice). Or: log grows slower than any positive power of ρ, so $u \in L^2(B_1(0))$ (integrand bounded).

2°. $\nabla u \in L^2(B_1(0))$:

$$\nabla u = \frac{1}{\log(1+1/|x|)}\cdot\frac{1}{1+1/|x|}\cdot\Bigl(-\frac{1}{|x|^2}\Bigr)\cdot\frac{x}{|x|}.$$

Thus

$$\begin{aligned}\int_{B_1(0)} |\nabla u|^2\,dx &\le C\int_0^1 \frac{1}{\log^2(1+1/\rho)\,(1+1/\rho)^2\rho^3}\,d\rho \\ &\le C\int_0^1 \frac{1}{\log^2(1+1/\rho)}\,\frac{1}{\rho}\,d\rho.\end{aligned}$$

Set $s = \log(1+1/\rho)$, so $1/\rho = e^s - 1$ and $\rho : 0 \to 1$ gives $s : \infty \to \log 2$. Then

$$I \le C\int_{\log 2}^\infty \frac{1}{s^2}\Bigl(\frac{1}{e^s-1}+1\Bigr)\,ds \le C\int_{\log 2}^\infty \frac{1}{s^2}\,ds < +\infty.$$

(The $p = n$ case is complicated!)

13.4.3 The case $p > n$

Definition 13.15 (Hölder spaces $C^{k,\gamma}(\Omega)$)**.** • $\|u\|_{C(\Omega)} := \sup_{x\in\Omega}|u(x)| = \|u\|_{L^\infty(\Omega)}$.

- γ-th Hölder seminorm ($\gamma \in (0,1]$):

$$[u]_{C^{0,\gamma}(\Omega)} := \sup_{x\ne y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

- γ-th Hölder norm:

$$\|u\|_{C^{0,\gamma}(\Omega)} := \|u\|_{C(\Omega)} + [u]_{C^{0,\gamma}(\Omega)}.$$

Remark 13.16. What does $[u]_{C^{0,\gamma}(\Omega)}$ describe?

Example 13.17 ($\gamma = 1$: Lipschitz)**.** If $|u(x)-u(y)| \le L|x-y|$ for all $x, y \in \Omega$, then $[u]_{C^{0,1}} \le L$. Lipschitz functions need not be differentiable everywhere (e.g. $f(x) = |x|$ is not differentiable at $x = 0$).

Example 13.18 ($C^1(\Omega) \subseteq C^{0,1}(\Omega)$ if Ω is bounded)**.**

$$\sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|} \leq C \sup_{x \in \Omega} |f'(x)|.$$

Locally: in a neighborhood of x_0, $f(x)$ is contained between $\pm|x|$.

Example 13.19 ($\gamma \in (0,1)$)**.** $|u(x) - u(y)| \leq C|x-y|^\gamma$. Locally $f(x)$ is contained between $\pm|x|^\gamma$.

Inclusions:

$$C^1(\Omega) \subseteq C^{0,1}(\Omega) \subseteq C^{0,\gamma}(\Omega),$$

since

$$\frac{|f(x)-f(y)|}{|x-y|^\gamma} \leq \frac{|f(x)-f(y)|}{|x-y|} \cdot |x-y|^{1-\gamma} \leq \frac{|f(x)-f(y)|}{|x-y|} \cdot \mathrm{diam}(\Omega)^{1-\gamma}.$$

Definition 13.20.

$$\|u\|_{C^{k,\gamma}(\Omega)} = \|u\|_{C^k(\Omega)} + \sum_{|\alpha|=k} [\partial^\alpha u]_{C^{0,\gamma}(\Omega)}.$$

Informally $C^{k,\gamma}(\Omega)$" $\approx$ "$W^{k+\gamma,\infty}(\Omega)$ (fractional derivatives).

Theorem 13.21 (Morrey's estimate)**.** *Assume $n < p \leq \infty$. Then there exists $C > 0$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \qquad \gamma = 1 - \frac{n}{p}.$$

In particular ($p = \infty$): $W^{1,\infty}(\mathbb{R}^n) \subset C^{0,1}(\mathbb{R}^n)$.

Proof. Step 1. We show

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - u(x)|\, dy \leq C \int_{B_r(x)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy$$

for every $B_r(x) \subseteq \mathbb{R}^n$. Fix $w \in \partial B_1(0)$. By FTC,

$$u(x + sw) - u(x) = \int_0^s Du(x + tw) \cdot w\, dt,$$

so $|u(x+sw) - u(x)| \leq \int_0^s |Du(x+tw)|\, dt$. Integrating over $\partial B_1(0)$,

$$\begin{aligned}
\int_{\partial B_1(0)} |u(x+sw) - u(x)|\, d\sigma_w &= \frac{1}{s^{n-1}} \int_{\partial B_s(x)} |u(y) - u(x)|\, d\sigma_y \\
&\leq \int_{\partial B_1(0)} \int_0^s |Du(x+tw)|\, dt\, d\sigma_w \\
&= \int_0^s \int_{\partial B_t(x)} \frac{|Du(y)|}{t^{n-1}}\, d\sigma_y\, dt = \int_{B_s(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy.
\end{aligned}$$

Hence

$$\int_{\partial B_s(x)} |u(z) - u(x)|\, d\sigma_z \leq s^{n-1} \int_{B_r(x)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.$$

Integrate s from 0 to r:

$$\int_{B_r(x)} |u(y) - u(x)|\, dy \leq \frac{r^n}{n} \int_{B_r(x)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.$$

Step 2.

$$u(x) = \frac{1}{|B_1(x)|} \int_{B_1(x)} (u(x) - u(y))\, dy + \frac{1}{|B_1(x)|} \int_{B_1(x)} u(y)\, dy,$$

so

$$\begin{aligned}
|u(x)| &\le C \int_{B_1(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy + C\|u\|_{L^p(B_1(x))} \qquad \big(bounded domain\big) \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)} \left(\int_{B_1(x)} |x-y|^{-(n-1)p/(p-1)}\, dy \right)^{(p-1)/p} + \|u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

The radial integral

$$\int_0^1 \rho^{-(n-1)p/(p-1)} \rho^{n-1}\, d\rho = \int_0^1 \rho^{-(n-1)/(p-1)}\, d\rho = C\big[\rho^{(p-n)/(p-1)}\big]_0^1$$

converges when $p > n$. Thus $|u(x)| \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$ for all $x \in \mathbb{R}^n$, i.e. $\|u\|_{C(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$.

Steps for Hölder continuity. Let $r = |x-y|$ and $W = B_r(x) \cap B_r(y)$. Then

$$|u(x) - u(y)| \le \frac{1}{|W|} \int_W |u(x) - u(z)|\, dz + \frac{1}{|W|} \int_W |u(z) - u(y)|\, dz =: I_1 + I_2.$$

Now

$$\begin{aligned}
I_1 &\le C \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(x) - u(z)|\, dz \le C \int_{B_r(x)} \frac{|Du(z)|}{|x-z|^{n-1}}\, dz \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)} \left(\int_{B_r(x)} |x-z|^{-(n-1)p/(p-1)}\, dz \right)^{(p-1)/p} \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)}\, r^{(p-n)/p} = C\|Du\|_{L^p(\mathbb{R}^n)}\, r^{1-n/p}.
\end{aligned}$$

Similarly $I_2 \le C\|Du\|_{L^p(\mathbb{R}^n)}\, r^{1-n/p}$. Hence

$$|u(x) - u(y)| \le C|x-y|^{1-n/p}\|Du\|_{L^p(\mathbb{R}^n)},$$

so $[u]_{C^{0,1-n/p}} \le C\|Du\|_{L^p(\mathbb{R}^n)}$, and therefore

$$\|u\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

□

Corollary 13.22. *If Ω is bounded and $n < p \le \infty$, then*

$$\|u\|_{C^{0,1-n/p}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

(by extension).

13.4.4 General Sobolev inequalities

Summary of embeddings for $W^{k,p}$:

- $kp < n$: $W^{k,p} \hookrightarrow L^q$ with $q = np/(n-kp)$.
- $kp > n$: $W^{k,p} \hookrightarrow C^{k-[n/p]-1,\gamma}$, where

$$\gamma = \begin{cases} [n/p] + 1 - n/p & \text{if } n/p \text{ is not an integer,} \\ \text{any } \gamma < 1 & \text{if } n/p \text{ is an integer.} \end{cases}$$

- $kp = n$: $W^{k,p} \hookrightarrow \{u : \int e^{C|u|^q} < +\infty\}$ (Trudinger / exponential integrability; related to BMO).

14 Compact embeddings, Poincaré inequalities, and difference quotients

14.1 Compact embedding

By GNS, $W^{1,p}(U) \hookrightarrow L^q(U)$ for all $1 \le p < n$ and $q \in [1, p^*]$. Goal: $W^{1,p}(U) \hookrightarrow\hookrightarrow L^q(U)$ for all $1 \le q < p^*$.

Definition 14.1 (Compact embedding)**.** Let X, Y be Banach spaces with $X \subset Y$. We say X is compactly embedded in Y ($X \hookrightarrow\hookrightarrow Y$) if $X \hookrightarrow Y$ and every bounded subset of X is (pre)compact in Y.

Theorem 14.2 (Rellich–Kondrachov)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded with ∂U of class C^1, and $1 \le p < n$. Then*

$$W^{1,p}(U) \hookrightarrow\hookrightarrow L^q(U) \qquad \forall\, 1 \le q < p^*.$$

Remark 14.3. Since $p^* > p$ and $p^* \to \infty$ as $p \to n$, one has in particular $W^{1,p}(U) \hookrightarrow\hookrightarrow L^p(U)$.

14.2 Poincaré inequalities and difference quotients

14.2.1 Poincaré's inequalities

Theorem 14.4 (Poincaré's inequality)**.** *Let $U \subset \mathbb{R}^n$ be open, bounded, and connected, with ∂U of class C^1, and $1 \le p \le \infty$. Then*

$$\left\| u - \frac{1}{|U|} \int_U u\, dy \right\|_{L^p(U)} \le C \|Du\|_{L^p(U)} \qquad \forall\, u \in W^{1,p}(U),$$

where C depends on n, p, U.

Proof (by contradiction). If the claim fails, there exists $\{u_k\} \subset W^{1,p}(U)$ with

$$\left\| u_k - \frac{1}{|U|} \int_U u_k\, dx \right\|_{L^p(U)} > k \|Du_k\|_{L^p(U)}.$$

Define

$$v_k = \frac{u_k - \frac{1}{|U|}\int_U u_k\, dy}{\left\| u_k - \frac{1}{|U|}\int_U u_k\, dy \right\|_{L^p}}.$$

Then $\frac{1}{|U|}\int_U v_k\, dy = 0$, $\|v_k\|_{L^p} = 1$, and $\|Dv_k\|_{L^p} < 1/k$ (by the assumption). In particular $\{v_k\}$ is bounded in $W^{1,p}$. By Rellich–Kondrachov ($W^{1,p}(U) \subset\subset L^p(U)$), there exist a subsequence v_{k_j} and $v \in L^p(U)$ with $v_{k_j} \to v$ in $L^p(U)$.

Moreover

$$\left| \frac{1}{|U|} \int_U v\, dy \right| \le C \|v - v_{k_j}\|_{L^p(U)} \to 0$$

(by Hölder), so $\frac{1}{|U|}\int_U v\, dy = 0$. Also $\|v\|_{L^p(U)} = 1$.

For every $\phi \in C_c^\infty(U)$,

$$\begin{aligned}
\left| \int_U v\, \partial_{x_i} \phi\, dx \right| &= \lim_{j\to\infty} \left| \int_U v_{k_j}\, \partial_{x_i}\phi\, dx \right| = \lim_{j\to\infty} \left| -\int_U D_{x_i} v_{k_j}\, \phi\, dx \right| \\
&\le \lim_{j\to\infty} \frac{1}{k_j} \|\phi\|_{L^\infty(U)} = 0.
\end{aligned}$$

Hence $Dv = 0$ a.e., so $v \in W^{1,p}(U)$ with $Dv = 0$. Since U is connected, $v \equiv C$ a.e.; the mean-zero condition forces $v \equiv 0$, contradicting $\|v\|_{L^p} = 1$. ∎

Remark 14.5 (Case $n = 1$)**.** Suppose $u \in W^{1,p}((0,1))$ and $Du = 0$ a.e. in $(0,1)$. Prove u is constant a.e. Assume toward a contradiction

$$u(x) = \begin{cases} C_1 & x \in (0,1/2), \\ C_2 & x \in (1/2,1) \end{cases}$$

with $C_1 \neq C_2$. Choose $\phi \in C_c^\infty((0,1))$ with $\phi(1/2) \neq 0$. Then

$$\begin{aligned} 0 &= \int_0^1 Du\,\phi\,dx = -\int_0^1 u\,D\phi\,dx \\ &= -(C_1\phi(1/2) - C_1\phi(0)) - (C_2\phi(1) - C_2\phi(1/2)) = (C_2 - C_1)\phi(1/2), \end{aligned}$$

a contradiction.

Corollary 14.6 (Poincaré inequality for a ball)**.** *Assume* $1 \leq p \leq \infty$*. There exists* $C = C(n,p)$ *such that*

$$\left\| u - \frac{1}{|B(x,r)|}\int_{B(x,r)} u(z)\,dz \right\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))}$$

for every ball $B(x,r) \subset \mathbb{R}^n$ *and every* $u \in W^{1,p}(B(x,r))$*.*

Remark 14.7. Relative to the usual Poincaré inequality, the constant here depends on the domain $V = B(x,r)$.

Proof. Set $v(y) := u(x + ry)$ for $y \in B(0,1)$. Then $u \in W^{1,p}(B(x,r))$ implies $v \in W^{1,p}(B(0,1))$. By Poincaré on the unit ball (with C independent of x, r),

$$\left\| v - \frac{1}{|B(0,1)|}\int_{B(0,1)} v\,dy \right\|_{L^p(B(0,1))} \leq C\|Dv\|_{L^p(B(0,1))}.$$

The change of variables $z = x + ry$ and $D_y v(y) = r\,D_z u(z)$ produces the factor Cr. □

14.2.2 Difference quotients

Assume $u : U \to \mathbb{R}$ is locally summable and $V \subset\subset U$.

Definition 14.8. (i) The i-th difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}, \qquad i = 1, \dots, n,$$

for all $x \in V$ and $0 < |h| < \operatorname{dist}(V, \partial U)$ (ensures $x + he_i$ stays in U).

(ii) $D^h u := (D_1^h u, \dots, D_n^h u)$.

Theorem 14.9 (Difference quotients and weak derivatives)**.** *(i) Suppose* $1 \leq p < \infty$ *and* $u \in W^{1,p}(U)$*. Then for every* $V \subset\subset U$*,*

$$\|D_i^h u\|_{L^p(V)} \leq C\|Du\|_{L^p(U)}$$

for some $C > 0$ *and all* $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$*.*

(ii) Assume $1 < p < \infty$*,* $u \in L^p(U)$*, and there exists* $C > 0$ *such that*

$$\|D_i^h u\|_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$*. Then* $u \in W^{1,p}(V)$*.*

Proof. First assume u is smooth. Then

$$u(x+he_i)-u(x)=\int_0^1 \partial_{x_i}u(x+the_i)\cdot h\,dt,$$

so

$$|D_i^h u(x)|\le\int_0^1 |Du(x+the_i)|\,dt.$$

Hence (elementary inequality)

$$|D^h u|\le\sum_{i=1}^n\int_0^1 |Du(x+the_i)|\,dt,$$

$$|D^h u|^p\le C\sum_{i=1}^n\left(\int_0^1 |Du(x+the_i)|\,dt\right)^p\le C\sum_{i=1}^n\int_0^1 |Du(x+the_i)|^p\,dt$$

(by Jensen: if f is convex and γ is a probability measure, then $f(\int u\,d\gamma)\le\int f(u)\,d\gamma$). Consequently

$$\begin{aligned}\int_V |D^h u|^p\,dx&\le C\sum_{i=1}^n\int_0^1\int_V |Du(x+the_i)|^p\,dx\,dt\\&\le C\int_U |Du|^p\,dx\end{aligned}$$

(let $y=x+the_i$). For general u, use the local approximation theorem (approximate using this identity).

(ii). For $\phi\in C_c^\infty(V)$ and h small,

$$\int_V u\,D_i^h\phi\,dx=-\int_V (D_i^{-h}u)\,\phi\,dx$$

(discrete integration by parts). Since $\|D_i^h u\|_{L^p(V)}\le C$, one has $\sup_h\|D_i^{-h}u\|_{L^p(V)}<+\infty$. For $1<p<\infty$ (L^p reflexive), Kakutani's theorem yields $v_i\in L^p(V)$ and $h_k\to 0$ with $D_i^{-h_k}u\rightharpoonup v_i$ weakly in $L^p(V)$. (Kakutani: a Banach space E is reflexive iff the closed unit ball is weakly compact.)

Then for $\phi\in C_c^\infty(V)$,

$$\begin{aligned}\int_V u\,\partial_{x_i}\phi\,dx&=\lim_{h_k\to 0}\int_V u\,D_i^{h_k}\phi\,dx\\&=-\lim_{h_k\to 0}\int_V (D_i^{-h_k}u)\,\phi\,dx=-\int_V v_i\,\phi\,dx,\end{aligned}$$

where the first limit uses

$$\left|\int_U u\big(D_i^{h_k}\phi-\partial_{x_i}\phi\big)\,dx\right|\le\|u\|_{L^p}\|D_i^{h_k}\phi-\partial_{x_i}\phi\|_{L^{p'}}\to 0$$

(Lebesgue dominated convergence theorem). Thus $v_i=D_{x_i}u$. Since $u,v_i\in L^p(V)$, one has $u\in W^{1,p}(V)$. □

Part IV
Weak solutions and regularity

15 Weak solutions of second-order elliptic equations

15.1 Compact embedding: counterexamples

Let U be bounded in $\mathbb{R}^n$. Then

$$W^{1,p}(U)\hookrightarrow\hookrightarrow L^q(U),\qquad p<n,$$

if $1 \le q < p^* = \dfrac{np}{n-p}$.

Counterexamples.

°1. $U = \mathbb{R}^n$. Let

$$f(x) = \begin{cases} 0 & x \le 0, \\ x & 0 < x < 1, \\ 2 - x & 1 \le x \le 2, \\ 0 & x > 2, \end{cases} \qquad Df(x) = \begin{cases} 0 & x < 0, \\ 1 & 0 < x < 1, \\ -1 & 1 < x < 2, \\ 0 & x > 2. \end{cases}$$

Define $f_n(x) = f(x-n)$ (translation). Then $\|Df_n\|_{L^p(\mathbb{R}^n)} \le C$, so $\|f_n\|_{W^{1,p}(\mathbb{R}^n)} \le C$ is bounded, but f_n does not converge in $L^q(\mathbb{R}^n)$ because $\|f_n\|_{L^q(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)} \not\to 0$, while $f_n \rightharpoonup 0$.

°2. $q = p^* = \dfrac{np}{n-p}$, assuming $\operatorname{supp} u \subset B(0,r) \subset U$. Set $u_\varepsilon(x) := \varepsilon^{-n/p^*} u(x/\varepsilon)$. Then

$$\begin{aligned} \|u_\varepsilon\|_{L^{p^*}(U)} &= \|u\|_{L^{p^*}(U)}, \\ \|\nabla u_\varepsilon\|_{L^p(U)} &= \varepsilon^{n/p-1+n/p^*} \|\nabla u\|_{L^p(U)} = \|\nabla u\|_{L^p(U)}. \end{aligned}$$

Therefore $\|u_\varepsilon\|_{W^{1,p}(U)} \le C$ is bounded as $\varepsilon \to 0$, $u_\varepsilon \rightharpoonup 0$, but $\|u_\varepsilon\|_{L^{p^*}(U)} \not\to 0$. Note that

$$\|u_\varepsilon\|_{L^q(U)} = \varepsilon^{n/q-n/p^*} \|u\|_{L^q(U)} \xrightarrow{\varepsilon \to 0} 0 \quad \text{if } q < p^*.$$

15.2 L^2 theory for weak solutions

In this section we consider

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

with Ω a bounded domain.

Remark 15.1 (Recall)**.** Long time ago,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

energy methods gave uniqueness, but existence needs a weaker space and improved regularity.

Definition 15.2.

$$Lu = -\partial_i\big(a^{ij}(x)\partial_j u\big) + b^i(x)\partial_i u + c(x)u \tag{1}$$

or

$$Lu = -a^{ij}(x)\partial_{ij} u + b^i(x)\partial_i u + c(x)u. \tag{2}$$

Form (1) is called *divergence form*; form (2) is *non-divergence form.* (Einstein summation: repeated indices are summed.) Here $a^{ij}(x)$ is a matrix of measurable functions.

Uniform ellipticity.

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \lambda \le 1 \le \Lambda,$$

and $a^{ij}(x), b^i(x), c(x) \in L^\infty(\overline{\Omega})$.

Remark 15.3. ① If $a^{ij} = \delta_{ij}$ and $b^i \equiv c \equiv 0$, then $Lu = -\Delta u$. The operator $-\Delta$ is uniformly elliptic because $\delta_{ij}\xi_i\xi_j = |\xi|^2$, so $\lambda = \Lambda = 1$.

② Assume symmetry $a^{ij}(x) = a^{ji}(x)$. For example

$$\partial_{11} u + 2\partial_{12} u + \partial_{22} u = \partial_{11} u + \partial_{12} u + \partial_{21} u + \partial_{22} u.$$

③ If $a^{ij} \in C^1$ then L may be written in either divergence or non-divergence form. But if a^{ij} is not differentiable, these two forms are not equivalent.

④ In matrix form, $a^{ij}(x)\xi_i\xi_j = \langle \xi, A\xi \rangle$. So L is uniformly elliptic if and only if A is positive definite with lowest eigenvalue $\lambda_0(x) \geq \theta > 0$ for all x. (A is real symmetric, so it has n real eigenvalues.)

15.3 Existence of weak solutions

Definition 15.4 (Bilinear form)**.** For all $u, v \in H_0^1(\Omega)$,

$$B[u,v] := \int_\Omega \sum_{i,j} a^{ij}\partial_i u\, \partial_j v + \sum_{i=1}^n b^i \partial_{x_i} u\, v + c\, uv\, dx.$$

Here

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)} \quad (\text{closed under } \|\cdot\|_{H^1})$$

or

$$\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$$

(the boundary condition via Trace).

16 Lax–Milgram theory, Fredholm alternative, and H^2 regularity

16.1 Problem set

Exercise 16.1. Sobolev embedding ($n \geq 2$):

- $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ for $p < n$, $p^* = ?$
- $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\gamma}(\mathbb{R}^n)$ for $p > n$, $\gamma = ?$
- $n = 1$ case: $\|u\|_{L^\infty([a,b])} \leq C\|u\|_{W^{1,1}((a,b))}$?

Exercise 16.2. Let u be a solution to $\partial_t u = \Delta u + u^3$ in $\mathbb{R}^n$ ($n = 1, 2, 3$).

- $u_\lambda = \lambda^s u(\lambda^2 t, \lambda x)$ is again a solution. Determine $s = ?$ (scaling invariance of the equation)
- Determine what $\dot{H}^k$ space satisfies $\|u_\lambda\|_{\dot{H}^k} = \|u\|_{\dot{H}^k}$? (scaling invariance of the norm)

① $\partial_t u_\lambda \sim \lambda^{s+2}\partial_t u$, $\Delta u_\lambda \sim \lambda^{s+2}\Delta u$, and $u_\lambda^3 \sim \lambda^{3s}u^3$. Hence $s + 2 = 3s \Rightarrow s = 1$, so $u_\lambda = \lambda u(\lambda^2 t, \lambda x)$.

②

$$\|u_\lambda\|_{\dot{H}^k} = \lambda \cdot \lambda^k \cdot \lambda^{-n/2}\|u\|_{\dot{H}^k} = \lambda^{1+k-n/2}\|u\|_{\dot{H}^k} \implies k = \frac{n}{2} - 1.$$

16.2 Application of Lax–Milgram

We now apply Lax–Milgram. Take

$$Lu = -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu.$$

In our case we want $B(u,v) = \langle f, v\rangle$ for all $v \in H_0^1(\Omega)$.

① $|B(u,v)| \leq C\|u\|_{H^1}\|v\|_{H^1}$ (boundedness).

② Not positive-definite! ($c < 0$ case). Instead we have, for some $\gamma > 0$,

$$\frac{\lambda}{2}\int_\Omega \left(|\nabla u|^2 + |u|^2\right) \leq B(u,u) + \gamma\|u\|_{L^2}^2 \tag{$*$}$$

hence

$$B(u,u) + \gamma\|u\|_{L^2}^2 \geq \frac{\lambda}{2}\|u\|_{H^1}^2.$$

Theorem 16.3 (First existence theorem)**.** *There exists $\gamma \ge 0$ such that for all $\mu \ge \gamma$ and all $f \in H^{-1}(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ to*

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Proof. Define $B_\mu(u,v) := B(u,v) + \mu\langle u, v\rangle$. Check:

① $B_\mu(u,v)$ is bilinear;

② $|B_\mu(u,v)| \le |B(u,v)| + \mu|\langle u,v\rangle| \le C\|u\|_{H^1}\|v\|_{H^1}$;

③ by $(*)$,

$$B_\mu(u,u) = B(u,u) + \mu(u,u) \ge B(u,u) + \gamma\|u\|_{L^2}^2 \ge \frac{\lambda}{2}\int_\Omega \big(|\nabla u|^2 + |u|^2\big) = \frac{\lambda}{2}\|u\|_{H^1}^2,$$

so $B_\mu(u,u)$ is positive definite.

Then Lax–Milgram applies. □

Restricting the variational equation $B[u,v] = \langle f, v\rangle$ to a finite-dimensional subspace of $H_0^1(\Omega)$ is the Galerkin method; piecewise-linear subspaces are assembled in Section 19.

Remark 16.4 (Example)**.** Let $\Omega = (0,\pi)$ and $Lu = -\dfrac{d^2}{dx^2}u + cu$. Then $u_k = \sin kx \Rightarrow Lu_k = (k^2 + c)u_k$. If $c = -k_0^2$ for some k_0, then $Lu_{k_0} = 0$.

- $B(u,v)$ is not positive definite: $B(u_{k_0}, u_{k_0}) = 0$.
- Therefore Lax–Milgram cannot apply. No uniqueness: if $Lu = 0$ then $L(u + u_{k_0}) = 0$.
- No existence for some f! For $f = u_{k_0}$: if there exists u with $Lu = u_{k_0}$, then

$$\|u_{k_0}\|_{L^2}^2 = (Lu, u_{k_0}) = (u, Lu_{k_0}) = 0,$$

  a contradiction.

16.3 Fredholm alternative

Question: what happens without adding μu?

Formal adjoint. E.g. $Lu = -\dfrac{d^2}{dx^2}u - u$ on $(0,\pi)$. Then $Lu = f$ has no solution if $\|f\|_{L^2}^2 = (Lu, f) = (u, L^*f)$ forces a contradiction whenever f is not orthogonal to $\ker L^*$ (here $L^* = L$ and $\ker L = \operatorname{span}\{\sin x\}$). Here L^* is the formal adjoint of L.

For divergence form

$$Lu = -\partial_i(a^{ij}\partial_j u) + b^i\partial_i u + cu,$$

$$\begin{aligned} (Lu, v) &= \int a^{ij}\partial_j u\,\partial_i v + b^i\partial_i u\, v + cuv\,dx \\ &= \int a^{ij}\partial_i v\,\partial_j u \boxed{-u\,\partial_i(b^i v)} + cuv\,dx =: (u, L^*v). \end{aligned}$$

(The first term is symmetric.) Therefore

$$L^*v = -\partial_i\big(a^{ij}\partial_j v + b^i v\big) + cv.$$

Definition 16.5 (Compact operators)**.** Let X, Y be Banach spaces. A bounded operator $K : X \to Y$ is compact if any bounded sequence $\{x_k\} \subset X$ has a subsequence $\{x_{k_j}\}$ such that Kx_{k_j} converges in Y.

Remark 16.6. E.g. $\mathrm{id} : W^{1,p}(\Omega) \to L^q(\Omega)$ for $q < \dfrac{np}{n-p}$ is compact. For an adjoint operator $K : H \to H$, $\langle Kx, y\rangle = \langle x, K^*y\rangle$.

Theorem 16.7 (Fredholm alternative)**.** *Let H be a Hilbert space and $K : H \to H$ a compact linear operator. Then*

(i) $\dim N(I-K) = \dim N(I-K^*) < +\infty$*;*

(ii) $R(I-K)$ *is closed and* $R(I-K) = N(I-K^*)^\perp$*.*

In particular, $R(I-K) = H$ if and only if $N(I-K^*) = \{0\}$. Either $(I-K)u = v$ has a unique solution for all $v \in H$ and $(I-K)^{-1}$ is bounded, or $(I-K)u = 0$ has a nontrivial solution and $(I-K)u = v$ has a solution if and only if $(v, w) = 0$ for all w with $w - K^*w = 0$.

Theorem 16.8 (Second existence theorem)**.** *Assume Ω is bounded, $a^{ij}, b^i, c \in L^\infty(\Omega)$, and L is uniformly elliptic. Exactly one of the following holds:*

- *Either*
$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{31}$$
has a unique weak solution $u \in H_0^1(\Omega)$ for all $f \in L^2$, and L^{-1} is bounded;
- *or*
$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{32}$$
has a nonzero weak solution, and
$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{33}$$
has a solution if and only if $(f, v) = 0$ for all $v \in H_0^1(\Omega)$ with $L^*v = 0$.

Proof. Let γ be from the first existence theorem. We know $(L+\gamma)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is a bounded linear operator, because $Lu + \gamma u = g$ in Ω can be solved uniquely. Now since $Lu = f$ if and only if $Lu + \gamma u = f + \gamma u$,

$$u = (L+\gamma)^{-1}(\gamma u + f), \qquad \text{i.e.} \qquad u - \gamma(L+\gamma)^{-1}u = (L+\gamma)^{-1}f. \tag{$*$}$$

Let $K = \gamma(L+\gamma)^{-1}$ and $\tilde{f} = (L+\gamma)^{-1}f$. Taking $H = L^2$, we have $K : L^2(\Omega) \to H_0^1(\Omega)$ bounded linear. We check K is compact as a map $L^2(\Omega) \to L^2(\Omega)$:

$$\tilde{K} = \mathrm{id}_{H_0^1(\Omega) \hookrightarrow L^2(\Omega)} \circ K_{L^2(\Omega) \to H_0^1(\Omega)}$$

is compact (composition of a compact embedding and a bounded map). Now K is compact, so by the Fredholm alternative theorem: either $(*)$ has a solution $u \in H$ for all $\tilde{f} \in H$, or $(*)$ has a nonzero solution for $f = 0$, and $(*)$ has a solution if and only if $(\tilde{f}, v) = 0$ for all $v \in N(I-K^*)$, i.e. $L^*v = 0$. $\square$

16.4 Regularity of weak solutions

16.4.1 H^2-regularity

Let u be a weak solution to $-\Delta u = f$ in Ω. Formally

$$\int_\Omega f^2 = \int_\Omega (-\Delta u)^2 = \int \sum \partial_{ii}u\,\partial_{jj}u = -\sum \int \partial_i u\,\partial_{ijj}u = \sum \int \partial_{ij}u\,\partial_{ij}u.$$

So if $u \in H^2(\Omega)$ we have $\sum \|\partial_{ij}u\|_{L^2} \le \|f\|_{L^2}$. The problem is to prove $u \in H^2(\Omega)$ (for general L), where

$$Lu = -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu.$$

Theorem 16.9 (Interior regularity)**.** *Assume $a^{ij} \in C^1(\Omega)$, $b^i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Let u be a weak solution to $Lu = f$ in Ω. Then $u \in H^2_{\mathrm{loc}}$, i.e. $u \in H^2(V)$ for all $V \subset\subset \Omega$, and*

$$\|u\|_{H^2(V)} \le C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

Remark 16.10. We can show $Lu = f$ a.e. in Ω (almost everywhere). Indeed $B(u, v) = (f, v)$ for all $v \in C_c^\infty(\Omega)$, where

$$B(u, v) := \int_\Omega a^{ij}\partial_i u\,\partial_j v + b^i\partial_i u \cdot v + cuv\,dx.$$

Since $u \in H^2_{\mathrm{loc}}$, we can integrate by parts ($v \in C_c^\infty(\Omega)$):

$$B(u, v) = \int_\Omega \big(-\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu\big)v\,dx = (Lu, v).$$

Hence $(Lu, v) = (f, v)$ for all $v \in C_c^\infty(\Omega)$, so $Lu = f$ a.e.

Remark 16.11. No boundary conditions are needed here (interior).

Remark 16.12. The RHS term $\|u\|_{L^2(\Omega)}$ is necessary: e.g. $u \equiv$ a constant and $L = -\Delta$, $f = 0$.

Proof. Take a cut-off η with $\eta = 1$ on V and $\eta = 0$ on W^c, where $V \subset\subset W \subset\subset \Omega$. The weak solution satisfies $B(u, v) = \langle f, v\rangle$ for all $v \in C_c^\infty(\Omega)$, so

$$\int_\Omega a^{ij}\partial_i u\,\partial_j v = \int \tilde f\,v = \int \big[f - \sum b^i\partial_i u - cu\big]v.$$

Note that since $u \in H^1(\Omega)$, we have $\tilde f \in L^2(\Omega)$. Let $|h| > 0$ be small and $k \in \{1, \dots, n\}$. Approximate $\partial_k u$ by defining

$$v := -D_k^{-h}(\eta^2 D_k^h u) \in H_0^1(\Omega)$$

(why H_0^1?), where

$$D_k^h u = \frac{u(x + he_k) - u(x)}{h}$$

is the difference quotient. Then

$$\begin{aligned}\int a^{ij}\partial_i u\,\partial_j v &= -\int a^{ij}\partial_i u\,\partial_j\big(D_k^{-h}(\eta^2 D_k^h u)\big)\\ &= \int D_k^h(a^{ij}\partial_i u)\,\partial_j(\eta^2 D_k^h u)\end{aligned}$$

(by the summation-by-parts property of difference quotients: $\int_\Omega v\,D_k^{-h}u = -\int_\Omega D_k^h v{\cdot}u$). Expanding,

$$\begin{aligned}&= \int_\Omega a^{ij}D_k^h(\partial_i u)\,\partial_j(\eta^2 D_k^h u) + \int_\Omega D_k^h(a^{ij})\,\partial_i u\,\partial_j(\eta^2 D_k^h u)\\ &= \int_\Omega \big[a^{ij}D_k^h(\partial_i u) + D_k^h(a^{ij})\partial_i u\big]\big[\eta^2 D_k^h\partial_j u + 2\eta\,\partial_j\eta\,D_k^h u\big]\\ &= \underbrace{\int_\Omega a^{ij}(D_k^h\partial_i u)(D_k^h\partial_j u)\,\eta^2}_{I_1} + I_2 + I_3 + I_4,\end{aligned}$$

where (writing $\xi_i = D_k^h \partial_i u$, $\xi_j = D_k^h \partial_j u$)

$$I_2 = \int_\Omega a^{ij} D_k^h(\partial_i u) \cdot 2\eta\, \partial_j \eta\, D_k^h u,$$
$$I_3 + I_4 = \int_\Omega D_k^h(a^{ij})\, \partial_i u\big(\eta^2 D_k^h \partial_j u + 2\eta\, \partial_j \eta\, D_k^h u\big).$$

By uniform ellipticity, $I_1 \geq \lambda \int \eta^2 |D_k^h Du|^2$. The rest:

$$|I_2| \leq C\|D_k^h Du \cdot \eta\|_2 \|\nabla \eta\, D_k^h u\|_2 \leq \varepsilon \int_\Omega |D_k^h Du \cdot \eta|^2 + \frac{C}{\varepsilon}\int |\nabla \eta|^2 |D_k^h u|^2,$$
$$|I_3| \leq \varepsilon \int_\Omega (|D_k^h Du|^2 \eta^2) + \frac{C}{\varepsilon}\int |Du|^2 \eta^2,$$
$$|I_4| \leq C\int |Du|^2 \eta^2 + C\int |D_k^h u|^2 \eta^2.$$

Therefore

$$\Big|\int_\Omega a^{ij} \partial_i u\, \partial_j v\Big| \geq \lambda \int_\Omega \eta^2 |D_k^h Du|^2 - 2\varepsilon \int_\Omega \eta^2 |D_k^h Du|^2 - \frac{C}{\varepsilon}\int_\Omega (|Du|^2 + |D_k^h u|^2)\eta^2.$$

Take $\varepsilon = \lambda/4$; also note $\|D_k^h u\|_2 \leq C\|Du\|_2$. Then

$$\int_\Omega a^{ij} \partial_i u\, \partial_j v\, dx \geq \frac{\lambda}{2}\int_\Omega \eta^2 |D_k^h Du|^2 - C\int_\Omega |Du|^2.$$

Using $\int_\Omega a^{ij} \partial_i u\, \partial_j v = \langle f, v\rangle$,

$$\frac{\lambda}{2}\int_\Omega \eta^2 |D_k^h Du|^2 \leq C\int_\Omega \big(f^2 + u^2 + |Du|^2\big)\, dx.$$

Hence $\int_V |D_k^h Du|^2$ is controlled, so $\sup_h \|D_k^h Du\|_{L^2(V)} < +\infty$ (this bounds $D^2 u$). Therefore there exists $\tilde{v} \in L^2(V)$ such that $D_k^{h_k} Du \rightharpoonup \tilde{v}$ weakly in $L^2(V)$. Thus $u \in H^2(V)$ and

$$\|u\|_{H^2(V)} \leq C\big(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\big).$$

(The H^1 term can be simplified to L^2: testing $v = \eta u$ gives $\|Du\|_{L^2(V)} \leq \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$.) □

Corollary 16.13 (Higher interior regularity)**.** *If $a^{ij}, b^i, c \in C^{m+1}(\Omega)$ and $f \in H^m(\Omega)$, then $u \in H^{m+2}_{\mathrm{loc}}(\Omega)$ and*

$$\|u\|_{H^{m+2}(V)} \leq C\big(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

17 Boundary regularity and De Giorgi local boundedness

17.1 Interior regularity (sketch)

Assume $a^{ij} \in C^1(\Omega)$, $b^i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Let u be a weak solution to $Lu = f$ in Ω. Then $u \in H^2_{\mathrm{loc}}$, i.e. $u \in H^2(V)$ for all $V \subset\subset \Omega$, and

$$\|u\|_{H^2(V)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

Idea. We show $\|u\|_{H^1(\Omega)} \le C(\|f\|_{L^2} + \|u\|_{L^2})$. To control $\|u\|_{H^1}$, test with $u \in H^1_0(\Omega)$:

$$B(u,u) = \langle f, u\rangle \implies \int a^{ij}\partial_i u\,\partial_j u + b^i\partial_i u\cdot u + cu^2 = \langle f, u\rangle,$$

hence

$$\begin{aligned}\int_\Omega a^{ij}\partial_i u\,\partial_j u &= \int_\Omega fu - b^i\partial_i u\cdot u - cu^2\,dx\\ &\implies \lambda\|\nabla u\|^2_{L^2} \le \|f\|^2_{L^2} + \|u\|^2_{L^2} + \|b\|_\infty\big(\varepsilon\|\nabla u\|^2_{L^2} + C(\varepsilon)\|u\|^2_{L^2}\big) + \|c\|_\infty\|u\|^2_{L^2}.\end{aligned}$$

Take $\varepsilon\|b\|_\infty = \lambda/2$ to obtain

$$\frac{\lambda}{2}\|\nabla u\|_2^2 \le \|f\|_2^2 + C\|u\|_2^2 \implies \|u\|^2_{H^1(\Omega)} \le C\big(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\big).$$

For $\|u\|^2_{H^2}$, try the test function $v = -\partial_k(\eta^2\partial_k u)$ (formally; otherwise use $D^h_k Du$). Why $v = \partial_k(\eta^2\partial_k u)$? Because $\partial_k u|_{\partial\Omega} \ne 0$ in general, so a cut-off is needed.

$$-\int a^{ij}\partial_i u\,\partial_j\big(\partial_k(\eta^2\partial_k u)\big) = \langle f - b^i\partial_i u - cu, v\rangle,$$

$$\begin{aligned}\text{LHS} &= \int a^{ij}\partial_{ik}u\,\partial_j(\eta^2\partial_k u) + \int \partial_k(a^{ij})\,\partial_i u\,\partial_j(\eta^2\partial_k u)\\ &= \int a^{ij}\partial_{ik}u\big(\partial_{jk}u\cdot\eta^2 + 2\eta\,\partial_j\eta\,\partial_k u\big)\\ &\quad+ \int \partial_k(a^{ij})\,\partial_i u\big(\partial_{jk}u\cdot\eta^2 + 2\eta\,\partial_j\eta\,\partial_k u\big).\end{aligned}$$

Rearranging,

$$\begin{aligned}\int a^{ij}\partial_{ik}u\,\partial_{jk}u\cdot\eta^2 &= \langle f - b^i\partial_i u - cu, v\rangle - \int 2a^{ij}\partial_{ik}u\,\eta\,\partial_j\eta\,\partial_k u\\ &\quad- \int \partial_k(a^{ij})\,\partial_i u\big(\partial_{jk}u\cdot\eta^2 + 2\eta\,\partial_j\eta\,\partial_k u\big),\end{aligned}$$

hence

$$\lambda\int |D^2u|^2\eta^2 \le \varepsilon\int |D^2u|^2\eta^2 + C\|u\|^2_{H^1} + C\|f\|^2_{L^2}.$$

Therefore

$$\|D^2u\|_{L^2(V)} \le C\big(\|f\|^2_{L^2} + \|u\|^2_{H^1_0}\big) \le C\big(\|f\|^2_{L^2} + \|u\|^2_{L^2}\big).$$

17.2 Boundary regularity

Theorem 17.1. *Let $a^{ij} \in C^1(\overline{\Omega})$, $b^i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Assume $\partial\Omega \in C^2$ and $u \in H^1_0(\Omega)$ is a weak solution. Then*

$$\|u\|_{H^2(\Omega)} \le C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

(H^2 of the whole Ω).

Sketch.

1) Use boundary straightening to the standard case (straightening): $x = \Psi(y)$, $y = \Phi(x)$. Then $a^{ij} \mapsto \tilde{a}^{kl}\ldots$ give new coefficients, still uniformly elliptic.

2) Model case: $\Omega = B(0,1) \cap \mathbb{R}^n_+$.

"Assume" $u \in H^2$ (otherwise use $D^h_k Du$).

- Tangential derivatives only ($l \neq n$): by a similarity argument,

$$\sum_{(l,m)\neq(n,n)} \|\partial_{lm} u\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}.$$

- For $\partial_{nn} u$:

$$a^{nn}\partial_{nn} u = -\sum_{(l,m)\neq(n,n)} a^{lm}\partial_{lm} u + (b^j - \partial_k a^{jk})\partial_j u + cu - f,$$

with $a^{nn}(x) \geq \lambda > 0$. Hence

$$|\partial_{nn} u| \leq C \sum_{(l,m)\neq(n,n)} |\partial_{lm} u| + |Du| + |u| + |f|,$$

and

$$\|\partial_{nn} u\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

Summary. If $Lu = f$ with weak solution $u \in H^1_0(\Omega)$, i.e.

$$\int_\Omega a^{ij}\partial_i u\, \partial_j v + b^i \partial_i u\, v + cuv = \int_\Omega fv \qquad \forall v \in H^1_0(\Omega),$$

we have: two existence theorems, and H^2-regularity. Recall: classical solutions need $u \in C^2(\Omega)$; there remains an $H^2 \to C^2$ gap.

17.3 Local boundedness

Theorem 17.2. *Suppose $a^{ij} \in L^\infty(B_1)$, $c \in L^q(B_1)$ for some $q > n/2$, with $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ and $\|a^{ij}\|_{L^\infty} + \|c\|_q \leq \Lambda$. Suppose $u \in H^1(B_1)$ is a subsolution ($b = 0$):*

$$\int_{B_1} a^{ij}\partial_i u\, \partial_j \varphi + cu\varphi \leq \int_{B_1} f\varphi \qquad \forall \varphi \in H^1_0(B_1),\ \varphi \geq 0.$$

If $f \in L^q(B_1)$ then $u^+ \in L^\infty_{\mathrm{loc}}(B_1)$ and for all $p > 0$,

$$\|u^+\|_{L^\infty(B_\theta)} \leq C\left\{ \frac{1}{(1-\theta)^\beta}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)} \right\}.$$

Method 1: De Giorgi. It suffices to consider the $\theta = \frac{1}{2}$ case:

$$\sup_{B_{1/2}} u^+ \leq C\big(2^{-n/2}\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\big).$$

Let $v = (u-k)^+$ for $k \geq 0$ (in the end let $k \to +\infty$), and let $\eta \in C^\infty_c(B_1)$ to be chosen later. Take $\varphi = v\eta^2$ as the test function. Note

$$Dv = \begin{cases} Du & \text{a.e. in } \{u > k\}, \\ 0 & \text{a.e. in } \{u \leq k\}. \end{cases}$$

Then

$$\begin{aligned} \int a^{ij} D_i u\, D_j \varphi &= \int a^{ij} D_i u\, D_j(v\eta^2) = \int a^{ij} D_i u\big(D_j v\, \eta^2 + 2\eta\, D_j \eta\, v\big) \\ &\geq \lambda \int_{\{u>k\}} |Dv|^2\eta^2 - 2\Lambda \int_{\{u>k\}} |Du|\,|D\eta|\, v\eta \\ &\geq \frac{\lambda}{2}\int_{\{u>k\}} |Dv|^2\eta^2 - \frac{2\Lambda^2}{\lambda}\int |D\eta|^2 v^2. \end{aligned}$$

By the subsolution inequality,

$$\int a^{ij} D_i u\, D_j(v\eta^2) + cu\, v\eta^2 \leq \int f\, v\eta^2,$$

and (using $u \leq v + k$)

$$\int |Du|^2\eta^2 \leq C\Big\{\int v^2|D\eta|^2 + |c|v^2\eta^2 + |c|k^2\eta^2 + |f|v\eta^2\, dx\Big\}.$$

Hence

$$\int |D(v\eta)|^2 \leq C\Big\{\int v^2|D\eta|^2 + \int |c|v^2\eta^2 + k^2\int |c|\eta^2 + \int |f|v\eta^2\Big\}. \tag{$\star$}$$

Recall by Sobolev embedding $W^{1,2} \hookrightarrow L^{2^*}$ with $2^* = \dfrac{2n}{n-2}$. Estimate the RHS of ($\star$) by Hölder: with $\dfrac{1}{q} + \dfrac{1}{2^*} + \dfrac{1}{q'} = 1$,

$$\begin{aligned}
\int |f|v\eta^2 &= \int_{\{v\eta\neq 0\}} |f|\, v\eta \cdot \eta \\
&\leq \|f\|_{L^q}\|v\eta\|_{L^{2^*}}\Big(\int_{\{v\eta\neq 0\}} \eta^{q'}\Big)^{1/q'} \\
&\leq \|f\|_{L^q}\|v\eta\|_{L^{2^*}}\,|\{v\eta\neq 0\}|^{1/2+1/n-1/q} \\
&\leq \varepsilon\|v\eta\|^2_{L^{2^*}} + C(\varepsilon)\|f\|^2_{L^q}\,|\{v\eta\neq 0\}|^{1+2/n-2/q} \\
&\leq \delta\|D(v\eta)\|^2_{L^2} + C(\delta)\|f\|^2_{L^q}\,|\{v\eta\neq 0\}|^{1-1/q+2/n}.
\end{aligned}$$

Note $1 + 2/n - 2/q > 1 - 1/q$ as long as $q > n/2$. Similarly

$$\begin{aligned}
\int |c|v^2\eta^2 &\leq \|c\|_{L^q}\|v\eta\|^2_{L^{2^*}}\,|\{v\eta\neq 0\}|^{2/n-1/q} \\
&\leq \|c\|_{L^q}\|D(v\eta)\|^2_{L^2}\,|\{v\eta\neq 0\}|^{2/n-1/q},
\end{aligned}$$

and

$$\int_{\{v\eta\neq 0\}} k^2|c|\eta^2 \leq k^2\|c\|_{L^q}\,|\{v\eta\neq 0\}|^{1-1/q}.$$

Combining (and using $2/n > 1/q$),

$$\begin{aligned}
\|D(v\eta)\|^2_{L^2} \leq C\bigg\{&\int v^2|D\eta|^2 + (k^2 + \|f\|^2_{L^q})\,|\{v\eta\neq 0\}|^{1-1/q} \\
&+ \|D(v\eta)\|^2_{L^2}\,|\{v\eta\neq 0\}|^{2/n-1/q}\bigg\}.
\end{aligned}$$

Now if $|\{v\eta\neq 0\}| < \varepsilon_0 \ll 1$, then $|\{v\eta\neq 0\}|^{2/n-1/q} < \varepsilon_0 \ll 1$, so

$$\|D(v\eta)\|^2_{L^2} \leq C\Big\{\int v^2|D\eta|^2 + (k^2 + \|f\|^2_{L^q})\,|\{v\eta\neq 0\}|^{1-1/q}\Big\}.$$

Again by Sobolev embedding and Hölder,

$$\begin{aligned}
\int (v\eta)^2 &\leq \Big(\int (v\eta)^{2^*}\Big)^{2/2^*}|\{v\eta\neq 0\}|^{1-2/2^*} = \|v\eta\|^2_{L^{2^*}}\,|\{v\eta\neq 0\}|^{2/n} \\
&\leq C\|D(v\eta)\|^2_{L^2}\,|\{v\eta\neq 0\}|^{2/n} \\
&\leq C\bigg(\int v^2|D\eta|^2\,dx\,|\{v\eta\neq 0\}|^{2/n} + (k^2 + \|f\|^2_{L^q})\,|\{v\eta\neq 0\}|^{1+2/n-1/q}\bigg).
\end{aligned}$$

Thus

$$\int (v\eta)^2 \le C\left[\int v^2|D\eta|^2 \,|\{v\eta\neq 0\}|^{2/n} + (k^2+\|f\|_{L^q}^2)\,|\{v\eta\neq 0\}|^{1+\varepsilon}\right]$$

for $\varepsilon = 2/n - 1/q$, if $|\{v\eta \neq 0\}|$ is small.

Now choose $\eta \in C_0^\infty(B_R)$ with $\eta = 1$ on B_r, $\eta = 0$ on B_R^c, and $|D\eta| \le \dfrac{2}{R-r}$. Set $A(k,r) := \{x \in B_r : u \ge k\}$ (so $\{v\eta \neq 0\} \subseteq A(k,R)$). Then for $0 < r < R < 1$ and $k > 0$,

$$\begin{aligned}\int_{A(k,r)} (u-k)^2 &\le \int v^2\eta^2 \\ &\le C\left[\frac{1}{(R-r)^2}\int v^2\,dx\,|\{v\eta\neq 0\}|^{\varepsilon} + (k^2+\|f\|_q^2)\,|\{v\eta\neq 0\}|^{1+\varepsilon}\right] \\ &\le C\left[\frac{1}{(R-r)^2}\int_{A(k,R)} (u-k)^2\,dx\,|A(k,R)|^{\varepsilon} + (k^2+\|f\|_q^2)\,|A(k,R)|^{1+\varepsilon}\right],\end{aligned}$$

provided $|A(k,R)|$ is small. Also

$$|A(k,R)| \le \frac{1}{k}\int_{A(k,R)} u^+ \le \frac{1}{k}\|u^+\|_{L^2},$$

so $|A(k,R)|$ is small if $k \ge k_0$ for some k_0.

Take any $h > k \ge k_0$ and $0 < r < R$ such that $A(k,r) \supset A(h,r)$ and $\int_{A(h,r)}(u-h)^2 \le \int_{A(k,r)}(u-k)^2$, and

$$|A(h,r)| \le |\{u-k > h-k\}| \le \frac{1}{(h-k)^2}\int_{A(k,r)}(u-k)^2.$$

Therefore

$$\begin{aligned}\int_{A(h,r)} (u-h)^2 &\le C|A(h,R)|^{\varepsilon}\left[\frac{1}{(R-r)^2}\int_{A(h,R)}(u-h)^2 + (h^2+\|f\|_{L^q}^2)\,|A(h,R)|\right] \\ &\le \frac{C}{(h-k)^{2\varepsilon}}\left(\int_{A(k,R)}(u-k)^2\right)^{\varepsilon}\left[\frac{1}{(R-r)^2}\int_{A(k,R)}(u-k)^2\right. \\ &\qquad \left. + \frac{h^2+\|f\|_{L^q}^2}{(h-k)^2}\int_{A(k,R)}(u-k)^2\right] \\ &\le \frac{C}{(h-k)^{2\varepsilon}}\left(\int_{A(k,R)}(u-k)^2\right)^{1+\varepsilon}\left[\frac{1}{(R-r)^2} + \frac{h^2+\|f\|_{L^q}^2}{(h-k)^2}\right].\end{aligned}$$

Hence

$$\|(u-h)^+\|_{L^2(B_r)} \le C\left(\frac{1}{R-r} + \frac{h+\|f\|_{L^q}}{h-k}\right)\frac{\|(u-k)^+\|_{L^2(B_R)}^{1+\varepsilon}}{(h-k)^{\varepsilon}}.$$

Set $y(k,r) = \|(u-k)^+\|_{L^2(B_r)}$, $k_\ell = k_0 + k\big(1-\frac{1}{2^\ell}\big)$, $r_\ell = \dfrac{1}{2} + \dfrac{1}{2^{\ell+1}}$, so $k_\ell - k_{\ell-1} = \dfrac{k}{2^\ell}$ and $r_{\ell-1} - r_\ell = \dfrac{1}{2^{\ell+1}}$. Setting $R = r_{\ell-1}$, $r = r_\ell$,

$$y(k_\ell, r_\ell) \le C\left[2^{\ell+1} + \frac{2^\ell}{k}\Big(k_0 + k\big(1-\tfrac{1}{2^\ell}\big) + \|f\|_{L^q}\Big)\right]\cdot\frac{2^{\ell\varepsilon}}{k^\varepsilon}\big(y(k_{\ell-1}, r_{\ell-1})\big)^{1+\varepsilon}. \tag{$**$}$$

Inductively one shows

$$y(k_\ell, r_\ell) \le \frac{y(k_0,r_0)}{\gamma^\ell} \qquad \text{for some } \gamma > 1.$$

Then

$$\big(y(k_{\ell-1}, r_{\ell-1})\big)^{1+\varepsilon} \le \left(\frac{y(k_0,r_0)}{\gamma^{\ell-1}}\right)^{1+\varepsilon} = \frac{y(k_0,r_0)^{\varepsilon}}{\gamma^{(\ell-1)(1+\varepsilon)}} \cdot \frac{y(k_0,r_0)}{\gamma^{\ell-1}}.$$

By $(**)$,

$$y(k_\ell, r_\ell) \le \frac{C\gamma^{1+\varepsilon}(k_0 + k + \|f\|_q)}{k^{1+\varepsilon}} \, 2^{\ell\varepsilon} \cdot \frac{y(k_{\ell-1}, r_{\ell-1})^{\varepsilon}}{\gamma^{\varepsilon\ell}} \cdot \frac{y(k_0,r_0)}{\gamma^{\ell}}.$$

We want

$$C\gamma^{1+\varepsilon}\left(\frac{k_0 + k + \|f\|_q}{k^{1+\varepsilon}}\right)\left(2^{\ell(1+\varepsilon)}\frac{y(k_0,r_0)^{\varepsilon}}{\gamma^{\varepsilon\ell}}\right) < 1.$$

Take $\gamma^{\varepsilon} = 2^{1+\varepsilon}$ $(\gamma > 1)$. Then it suffices that

$$C\gamma^{1+\varepsilon}\left(\frac{y(k_0,r_0)}{k}\right)^{\varepsilon}\frac{k_0 + k + \|f\|_q}{k} < 1,$$

which holds by taking k large. Now

$$y(k_\ell, r_\ell) \le \frac{y(k_0,r_0)}{\gamma^{\ell}} \longrightarrow 0 \quad \text{as } \ell \to \infty,$$

and $k_\ell \to k_0 + k$, $r_\ell \to \frac{1}{2}$, i.e. $y(k_0 + k, \frac{1}{2}) = 0$. Thus

$$\big\|(u - (k_0 + k))^+\big\|_{L^2(B_{1/2})} = 0 \implies u^+ \le k_0 + k \quad \text{in } B_{1/2},$$

so

$$\sup_{B_{1/2}} u^+ \le C\big\{k_0 + \|f\|_{L^q} + y(k_0,r_0)\big\},$$

where $k_0 + \|f\|_{L^q}$ and $y(k_0,r_0)$ are both controlled by $C\|u^+\|_{L^2(B_1)}$. □

18 Nash–Moser iteration, comparison principles, and moving planes

18.1 Method 2 (Nash–Moser)

Assume $b^i = c = f = 0$. The idea is to show the reverse Hölder inequality

$$\|u\|_{L^p(B_{r_1})} \le C\|u\|_{L^{p_0}(B_{r_2})} \qquad \text{for } p > p_0,\ r_1 < r_2$$

(issue: $C \sim \dfrac{1}{r_2 - r_1}$).

Step 1.

For $m > 0$ define

$$\bar{u}_m = \begin{cases} u^+ & \text{if } u < m, \\ m & \text{if } u \ge m \end{cases}$$

(will let $m \to +\infty$ later). Then $D\bar{u}_m = 0$ in $\{u < 0\}$ (where $u^+ = 0$) and in $\{u > m\}$ (where $\bar{u}_m \equiv m$ is constant). Also $\bar{u}_m \le u^+$. Set the test function

$$\varphi = \eta^2\big(\bar{u}_m^{\beta} u^+\big) \in H_0^1(B_1) \qquad \text{for some } \beta \ge 0,$$

with $\eta \ge 0 \in C_c^{\infty}(B_1)$. Then

$$D\varphi = 2\eta\, D\eta\, (\bar{u}_m^{\beta} u^+) + \eta^2\big(\beta\bar{u}_m^{\beta-1} D\bar{u}_m\big)u^+ + \eta^2\bar{u}_m^{\beta} Du^+.$$

Note $D\bar{u}_m \cdot u^+\, \bar{u}_m^{\beta-1} = D\bar{u}_m\, \bar{u}_m^{\beta}$ because $u^+ = \bar{u}_m$ whenever $D\bar{u}_m \ne 0$ (i.e. $0 \le u < m$). Hence

$$D\varphi = 2\eta\, D\eta\, (\bar{u}_m^{\beta} u^+) + \eta^2\bar{u}_m^{\beta}\big(\beta D\bar{u}_m + Du^+\big).$$

Now $\varphi = 0 = D\varphi$ in $\{u \le 0\}$ (where $u^+ = \bar u_m \equiv 0$), so we only integrate in $\{u > 0\}$ (and $D_i u = D_i u^+$ there):

$$\int a^{ij} D_i u\, D_j\varphi = \int a^{ij} D_i u\Big(2\eta\, D_j\eta\, \bar u_m^\beta u^+ + \eta^2 \bar u_m^\beta\big(\beta D_j \bar u_m + D_j u^+\big)\Big).$$

Note

$$D_i u^+\, D_j \bar u_m = \begin{cases} 0 & \text{if } u \ge m, \\ D_i \bar u_m\, D_j \bar u_m & \text{if } u < m. \end{cases}$$

Thus

$$\int a^{ij} D_i \bar u\, \eta^2 \bar u_m^\beta\, \beta\, D_j \bar u_m \ge \lambda\beta \int \eta^2 \bar u_m^\beta |D\bar u_m|^2,$$

and

$$\int a^{ij} D_i u^+\, \eta^2 \bar u_m^\beta\, D_j u^+ \ge \lambda \int \eta^2 \bar u_m^\beta |Du^+|^2.$$

For the remaining terms (writing $\bar u = u^+$),

$$\begin{aligned} \int a^{ij} D_i u^+\, D_j\eta\, (\bar u_m^\beta u^+)\, \eta &\le \Lambda \int |Du^+|\, |D\eta|\, \bar u_m^\beta u^+\, \eta \\ &\le \frac{\lambda}{2} \int \eta^2 \bar u_m^\beta |Du^+|^2 + \frac{2\Lambda^2}{\lambda} \int |D\eta|^2 \bar u_m^\beta (u^+)^2. \end{aligned}$$

Hence

$$\beta \int \eta^2 \bar u_m^\beta |D\bar u_m|^2 + \int \eta^2 \bar u_m^\beta |Du^+|^2 \le C \int |D\eta|^2 \bar u_m^\beta (u^+)^2.$$

Set $w = \bar u_m^{\beta/2} u^+$; then

$$|Dw|^2 \le (1+\beta)\big\{\beta \bar u_m^\beta |D\bar u_m|^2 + \bar u_m^\beta |Du^+|^2\big\},$$

so

$$\int |Dw|^2 \eta^2 \le C(1+\beta) \int w^2 |D\eta|^2 \qquad \text{or} \qquad \int |D(w\eta)|^2 \le C(1+\beta) \int w^2 |D\eta|^2.$$

Step 2.

By the Sobolev inequality,

$$\Big(\int |\eta w|^{2^*}\Big)^{2/2^*} \le C \int |D(\eta w)|^2 \le C(1+\beta) \int w^2 |D\eta|^2.$$

Choose $\eta = 1$ on B_r, $\eta = 0$ on B_R^c, with $|D\eta| \le \dfrac{2}{R-r}$. Then

$$\Big(\int_{B_r} |w|^{2^*}\Big)^{2/2^*} \le \frac{C(1+\beta)}{(R-r)^2} \int_{B_R} w^2.$$

By definition $w = \bar u_m^{\beta/2} u^+$, so

$$\Big(\int_{B_r} (u^+)^{2^*} \bar u_m^{\beta 2^*/2}\Big)^{2/2^*} \le \frac{C(1+\beta)}{(R-r)^2} \int_{B_R} (u^+)^2 \bar u_m^\beta.$$

Since $\bar u_m \le u^+$, set $\gamma = 2 + \beta$:

$$\Big(\int_{B_r} \bar u_m^{(\gamma/2)\, 2^*}\Big)^{2/2^*} \le \frac{C(\gamma-1)}{(R-r)^2} \int_{B_R} (u^+)^\gamma.$$

Let $m \to \infty$ and define $\chi = 2^*/2$:

$$\Big(\int_{B_r} \bar u^{\gamma\chi}\Big)^{1/\chi} \le \frac{C(\gamma-1)}{(R-r)^2} \int_{B_R} (u^+)^\gamma.$$

Therefore

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \le \left(C\frac{\gamma-1}{(R-r)^2}\right)^{1/\gamma} \|\bar{u}\|_{L^{\gamma}(B_R)}.$$

Step 3.

Iterate beginning with $\gamma = 2$:

$$\|\bar{u}\|_{L^{2\chi}(B_r)} \le \left(C\frac{1}{(R-r)^2}\right)^{1/2} \|\bar{u}\|_{L^{2}(B_R)}.$$

Also iterate the radii:

$$\gamma_i = 2\chi^i \quad (= 2,\, 2\chi,\, 2\chi^2, \dots), \qquad r_i = \frac{1}{2} + \frac{1}{2^{i+1}} \quad (r_i \to \tfrac{1}{2}),$$

with $r_{i-1} - r_i = \dfrac{1}{2^{i+1}}$. Then

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le \left(C\frac{\chi^i}{(r_{i-1}-r_i)^2}\right)^{1/\gamma_i} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})},$$

and the prefactor is bounded by a factor of order C^{i/χ^i}. Hence

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C^{\sum i/\chi^i} \|\bar{u}\|_{L^2(B_1)},$$

where the sum is bounded since $\chi = 2^*/2 = n/(n-2) > 1$. Letting $i \to +\infty$,

$$\sup_{B_{1/2}} \bar{u} \le C\|\bar{u}\|_{L^2(B_1)}, \qquad \text{or} \qquad \sup_{B_{1/2}} u^+ \le C\|u^+\|_{L^2(B_1)}.$$

18.2 Comparison principle for elliptic equations

Let Ω be open and bounded, and

$$Lu := a_{ij}D_{ij}u(x) + b_iD_iu(x) + cu(x) \ge 0.$$

- If $c \le 0$, then L is still a “negative” operator and

$$\max_{\overline{\Omega}} u^+ = \max_{\partial\Omega} u^+.$$

  (If $u(x_0) = \max_{\overline{\Omega}} u^+$ with $x_0 \in \mathrm{int}(\Omega)$, then $Lu(x_0) \le 0$, using $u(x_0) \ge 0$ and $c \le 0$.)

- If $\|c\|_\infty \le M$, require $u \le 0$. Then

$$(L-M)u = a_{ij}D_{ij}u + b_iD_iu + (c-M)u = Lu - Mu \ge -Mu \ge 0$$

  (since $Lu \ge 0$, $u \le 0$, and $c - M \le 0$). If $u \le 0$ then $u \equiv 0$ or $u < 0$.

Theorem 18.1 (Narrow domain maximum principle)**.** *Assume $\overline{\Omega} \subset \{0 < x_1 < d\}$ for $d \ll 1$, and*

$$\begin{cases} \Delta u + c(x)u \ge 0 & \text{in } \Omega, \\ u \le 0 & \text{on } \partial\Omega, \end{cases} \tag{34}$$

where $|c(x)| \le M$. Then $u < 0$ or $u \equiv 0$ in Ω.

Idea. In this domain, consider $w = \sin(\pi x_1/d)$. Then

$$\begin{cases} \Delta w + (\pi/d)^2 w = 0 & \text{in } \Omega, \\ w > 0 & \text{on } \overline{\Omega}, \end{cases}$$

and we only need $\Delta w + c(x)w \le 0$, which holds since $(\pi/d)^2 \gg 1$. □

Lemma 18.2. *(Ω need not be narrow.) Suppose $Lu \ge 0$ in Ω and*

$$\begin{cases} Lw \le 0 & \text{in } \Omega, \\ w > 0 & \text{on } \overline{\Omega}, \end{cases}$$

where $L := \Delta + c$ with $|c| \le M$. Then

$$\max_{\overline{\Omega}} \frac{u^+}{w} = \max_{\partial\Omega} \Big(\frac{u}{w}\Big)^+.$$

Proof. Write $v = u/w$ so $u = vw$. Then

$$\Delta u = \Delta v\, w + v\Delta w + 2Dv \cdot Dw,$$

hence

$$\Delta v\, w + v\Delta w + 2Dv \cdot Dw + cvw \ge 0,$$

i.e.

$$\Delta v + \frac{v}{w}(\Delta w + cw) + 2\frac{Dv}{w} \cdot Dw \ge 0.$$

Define

$$\tilde{L}v := \Delta v + 2Dv \cdot \frac{Dw}{w} + \tilde{c}\, v \ge 0 \quad \text{in } \Omega,$$

where $\tilde{c}(x) := \dfrac{\Delta w + cw}{w} \le 0$, and $v^+ = u^+/w$. Assume $v^+(x_0) = \max_{\overline{\Omega}} v^+$ with $x_0 \in \Omega$ and $v(x_0) > 0$. Then (after an "ε-trick")

$$\Delta v + 2Dv \cdot \frac{Dw}{w} + \tilde{c}\, v \le 0 \quad \text{at } x_0,$$

forcing $x_0 \in \partial\Omega$. □

Back to the narrow-domain case:

$$\max_{\overline{\Omega}} \Big(\frac{u}{w}\Big)^+ = 0,$$

and by Hopf's lemma, $u < 0$ in Ω or $u \equiv 0$.

18.3 Moving planes

Theorem 18.3. *Suppose $u \in C(\overline{B_1}) \cap C^2(B_1)$ and $u > 0$ in B_1, with*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{35}$$

where f is locally Lipschitz. Then u is radially symmetric in B_1 and $\dfrac{\partial u}{\partial r}(x) < 0$ for all $x \ne 0$.

Remark 18.4. In fact the symmetry claim only requires Ω bounded and convex in the x_1-direction and symmetric with respect to $\{x_1 = 0\}$. The proof uses the narrow-domain maximum principle and the small-volume maximum principle.

Exercise 18.5. Show that the conclusion that u is radially symmetric is false without the assumption that u is positive.

Lemma 18.6. *Let Ω be a bounded domain that is convex in the x_1-direction and symmetric with respect to the plane $\{x_1 = 0\}$. Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a positive solution of*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{36}$$

with f locally Lipschitz in $\mathbb{R}$. Then u is symmetric with respect to x_1 and

$$\frac{\partial u}{\partial x_1}(x) < 0 \qquad \text{for any } x \in \Omega \text{ with } x_1 > 0.$$

Proof of the ball theorem. The ball theorem follows immediately from the lemma: u is decreasing in any given radial direction (the unit ball is symmetric with respect to any plane through the origin), and u is invariant under reflection across any such hyperplane, hence radially symmetric. □

Proof of the lemma. Use coordinates $x = (x_1, y) \in \Omega$ with $y \in \mathbb{R}^{n-1}$. We prove

$$u(x_1, y) < u(x_1^*, y) \qquad \text{for all } x_1 > 0 \text{ and } -x_1 < x_1^* < x_1. \tag{4.3}$$

This implies monotonicity in x_1 for $x_1 > 0$. Letting $x_1^* \to -x_1$ gives $u(x_1, y) \le u(-x_1, y)$ for $x_1 > 0$; reversing the direction yields reflection symmetry $u(x_1, y) = u(-x_1, y)$.

To prove (4.3), given any $\lambda \in (0, a)$ with $a = \sup_\Omega x_1$, take the moving plane $T_\lambda = \{x_1 = \lambda\}$ and

$$\Sigma_\lambda = \{x \in \Omega : x_1 > \lambda\}.$$

Let $x_\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ be the reflection of x across T_λ, and set

$$w_\lambda(x) = u(x) - u(x_\lambda) \qquad \text{for } x \in \Sigma_\lambda.$$

By the mean value theorem,

$$\Delta w_\lambda = f\big(u(x_\lambda)\big) - f\big(u(x)\big) = -c(x, \lambda)\, w_\lambda \quad \text{in } \Sigma_\lambda,$$

where $|c(x, \lambda)| \le \mathrm{Lip}(f)$ (a recurring trick: the difference of two solutions of a semilinear equation satisfies a linear equation with bounded but unknown coefficient). The boundary $\partial\Sigma_\lambda$ consists of a piece of $\partial\Omega$ (where $w_\lambda = -u(x_\lambda) < 0$) and a part of T_λ (where $x = x_\lambda$ and $w_\lambda = 0$). Thus

$$\begin{cases} \Delta w_\lambda + c(x, \lambda) w_\lambda = 0 & \text{in } \Sigma_\lambda, \\ w_\lambda \le 0 \text{ and } w_\lambda \not\equiv 0 & \text{on } \partial\Sigma_\lambda. \end{cases} \tag{4.5}$$

By the narrow-domain maximum principle, $w_\lambda < 0$ in Σ_λ for λ close to a. Moreover, w_λ attains its maximum 0 over $\overline{\Sigma}_\lambda$ along T_λ, so by Hopf's lemma

$$\left.\frac{\partial w_\lambda}{\partial x_1}\right|_{x_1=\lambda} = 2 \left.\frac{\partial u}{\partial x_1}\right|_{x_1=\lambda} < 0.$$

Hence $\partial u / \partial x_1 < 0$ for $x_1 > 0$, once we know $w_\lambda < 0$ in Σ_λ for all $\lambda \in (0, a)$. Equivalently,

$$u(x_1, x') < u(2\lambda - x_1, x') \qquad \text{for all } \lambda \in (0, x_1).$$

Since $c(x, \lambda)$ has no fixed sign, the usual maximum principle fails. There exists $\delta_c > 0$ (depending only on $\|c\|_{L^\infty}$) such that the narrow-domain maximum principle holds for $Lu = \Delta u + c(x)u$ whenever the width in the x_1-direction is less than δ_c. Thus for $a - \delta_c < \lambda < a$, the domain Σ_λ is narrow enough that $w_\lambda < 0$ in Σ_λ (using $w_\lambda \le 0$ and $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$).

Now decrease λ (move T_λ left). Let (λ_0, a) be the largest interval such that $w_\lambda < 0$ in Σ_λ for all $\lambda \in (\lambda_0, a)$. If $\lambda_0 = 0$ we are done. Assume for contradiction that $\lambda_0 > 0$. By continuity, $w_{\lambda_0} \le 0$ in Σ_{λ_0}, and $w_{\lambda_0} \not\equiv 0$ on $\partial\Sigma_{\lambda_0}$. The strong maximum principle gives

$$w_{\lambda_0} < 0 \quad \text{in } \Sigma_{\lambda_0}. \tag{4.7}$$

We claim that for sufficiently small $\varepsilon > 0$,

$$w_{\lambda_0-\varepsilon} < 0 \quad \text{in } \Sigma_{\lambda_0-\varepsilon}, \tag{4.8}$$

contradicting maximality of λ_0 unless $\lambda_0 = 0$.

Choose a simply connected closed set $K \subset \Sigma_{\lambda_0}$ with smooth boundary, nearly all of Σ_{λ_0} in the sense that $|\Sigma_{\lambda_0} \setminus K| < \delta/2$. From (4.7) there exists $\eta > 0$ with $w_{\lambda_0} \le -\eta$ on K. By continuity,

$$w_{\lambda_0-\varepsilon} < -\frac{\eta}{2} < 0 \quad \text{on } K. \tag{4.9}$$

On the boundary of $\Sigma_{\lambda_0-\varepsilon} \setminus K$ we have $w_{\lambda_0-\varepsilon} \le 0$ on $\partial\Sigma_{\lambda_0-\varepsilon}$ and $w_{\lambda_0-\varepsilon} < 0$ on ∂K, hence $w_{\lambda_0-\varepsilon} \le 0$ on $\partial(\Sigma_{\lambda_0-\varepsilon} \setminus K)$. For ε small, $|\Sigma_{\lambda_0-\varepsilon} \setminus K| < \delta$. Choosing δ small enough (depending only on $\|c\|_{L^\infty(\Omega)}$), the maximum principle for domains of small volume yields $w_{\lambda_0-\varepsilon} \le 0$ in $\Sigma_{\lambda_0-\varepsilon} \setminus K$; the strong maximum principle upgrades this to < 0. Combined with (4.9) we obtain (4.8), a contradiction. Thus $\lambda_0 = 0$, completing the proof. □

19 Galerkin finite elements for elliptic problems

Finite differences replace derivatives by stencils on a Cartesian grid. Once the Poisson problem is recast in weak form (Dirichlet principle; Lax–Milgram), a different discretization is available: restrict the variational equation to a finite-dimensional subspace of H_0^1. This is the Galerkin method; piecewise polynomial subspaces on a mesh yield the finite element method (FEM). The construction works on general polygonal domains, treats Neumann data as a boundary integral, and inherits stability from the continuous bilinear form rather than from a discrete maximum principle.

19.1 Galerkin orthogonality and Céa's lemma

Let $V = H_0^1(\Omega)$ and let $a : V \times V \to \mathbb{R}$ be a bounded, coercive bilinear form,

$$|a(u,v)| \le M\|u\|_V\|v\|_V, \qquad a(v,v) \ge \alpha\|v\|_V^2, \qquad \alpha, M > 0.$$

Given $f \in V'$, Lax–Milgram yields a unique $u \in V$ with

$$a(u,v) = \langle f, v\rangle \qquad \forall v \in V. \tag{37}$$

For the Dirichlet Laplacian one has $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx$ and $\langle f, v\rangle = \int_\Omega f v\, dx$, so (37) is the Euler–Lagrange equation of the Dirichlet energy.

Definition 19.1 (Galerkin method)**.** Let $V_h \subset V$ be a finite-dimensional subspace. The Galerkin approximation $u_h \in V_h$ is defined by

$$a(u_h, v_h) = \langle f, v_h\rangle \qquad \forall v_h \in V_h. \tag{38}$$

Existence and uniqueness of u_h follow by applying Lax–Milgram on V_h (the same α, M work). Subtracting (38) from (37) with $v = v_h$ gives *Galerkin orthogonality*

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h.$$

Theorem 19.2 (Céa)**.**

$$\|u - u_h\|_V \le \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V.$$

Proof. For any $v_h \in V_h$, coercivity and orthogonality yield

$$\alpha\|u-u_h\|_V^2 \le a(u-u_h,u-u_h) = a(u-u_h,u-v_h) \le M\|u-u_h\|_V\|u-v_h\|_V.$$

Cancel $\|u-u_h\|_V$ and take the infimum over v_h. □

Thus the Galerkin error is controlled by the best-approximation error in V_h. Choosing V_h is a question of approximation theory; assembling (38) is a question of local bases.

19.2 Piecewise-linear elements in one dimension

Take $\Omega = (0,1)$, $V = H_0^1(0,1)$, and $a(u,v) = \int_0^1 u'v'\,dx$. Let $h = 1/N$, $x_j = jh$ for $j = 0,\dots,N$, and let φ_j be the hat function supported on $[x_{j-1},x_{j+1}]$ already used as a nodal interpolant in the finite-difference chapter:

$$\varphi_j(x) = \begin{cases} (x-x_{j-1})/h & x \in [x_{j-1},x_j], \\ (x_{j+1}-x)/h & x \in [x_j,x_{j+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$V_h = \operatorname{span}\{\varphi_1,\dots,\varphi_{N-1}\} = \big\{v_h \in C([0,1]) : v_h|_{[x_j,x_{j+1}]} \text{ is affine},\ v_h(0)=v_h(1)=0\big\}.$$

Write $u_h = \sum_{j=1}^{N-1} U_j\varphi_j$. Then (38) becomes the linear system $A\underline{U} = \underline{F}$ with

$$A_{ij} = a(\varphi_j,\varphi_i) = \int_0^1 \varphi_j'\varphi_i'\,dx, \qquad F_i = \int_0^1 f\varphi_i\,dx. \tag{39}$$

A direct computation on the two elements meeting at x_i yields the tridiagonal stiffness matrix

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}.$$

Thus A is a positive multiple of the finite-difference matrix B from the Poisson stencil. If the load is replaced by the lumped values $F_i \approx hf(x_i)$, the two schemes coincide. Consistent assembly of F_i (exact integration against φ_i, or a quadrature of sufficient order) keeps the Galerkin structure and the Céa bound.

Remark 19.3 (Neumann data)**.** If the boundary condition is $u'(1) = g$ rather than $u(1) = 0$, the space becomes $\{v \in H^1(0,1) : v(0) = 0\}$ and the load gains the boundary term $g\,v(1)$. No stencil modification at $x = N$ is required: the essential condition is built into V_h, the natural condition appears on the right-hand side.

19.3 Interpolation and a priori rates

Let $I_hu = \sum_{j=1}^{N-1} u(x_j)\varphi_j$ be the nodal interpolant (for $u \in H^2 \cap H_0^1$). On each element, Taylor expansion with integral remainder gives

$$\|u-I_hu\|_{L^2(0,1)} \le Ch^2|u|_{H^2}, \qquad \|(u-I_hu)'\|_{L^2(0,1)} \le Ch|u|_{H^2}.$$

Céa's lemma with $v_h = I_hu$ therefore yields

$$\|u-u_h\|_{H^1(0,1)} \le Ch|u|_{H^2(0,1)}. \tag{40}$$

The L^2 rate is one order higher. Let $w \in H_0^1$ solve the dual problem $a(v,w) = (u-u_h,v)_{L^2}$ for all $v \in V$. Elliptic regularity gives $|w|_{H^2} \le C\|u-u_h\|_{L^2}$. Galerkin orthogonality and (40) for w then yield the Aubin–Nitsche estimate

$$\begin{aligned}\|u-u_h\|_{L^2}^2 &= a(u-u_h,w) = a(u-u_h,w-I_hw) \\ &\le C\|u-u_h\|_{H^1}\,h|w|_{H^2} \le Ch\|u-u_h\|_{H^1}\|u-u_h\|_{L^2},\end{aligned}$$

hence

$$\|u - u_h\|_{L^2(0,1)} \le Ch^2 |u|_{H^2(0,1)}. \tag{41}$$

Example 19.4 (1D Poisson, P_1 elements)**.** For $-u'' = \sin(\pi x)$ on $(0,1)$ with $u(0) = u(1) = 0$, one has $u = \pi^{-2}\sin(\pi x)$. Figure 7 shows the P_1 solution at $N = 8$ and the H^1 / L^2 errors against h. The observed slopes match (40)–(41).

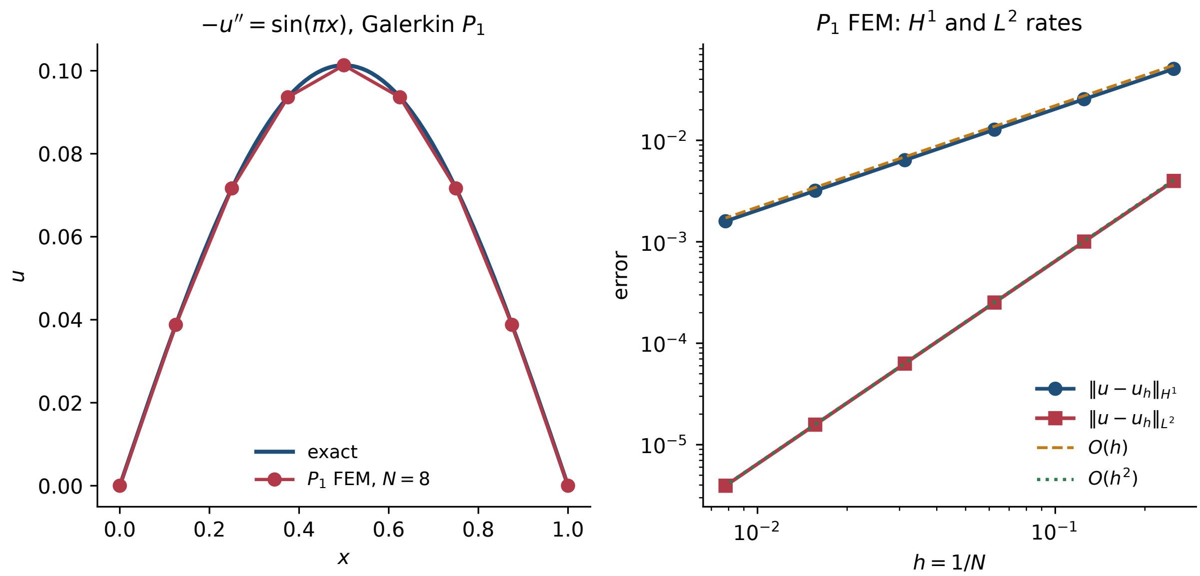


Figure 7: P_1 Galerkin for $-u'' = \sin(\pi x)$. Left: piecewise-linear solution versus the exact sine mode. Right: H^1 error $O(h)$ and L^2 error $O(h^2)$.

Exercise 19.5. Assemble A and F for $-u'' = 1$ on $(0,1)$ with $u(0) = u(1) = 0$ and $N = 4$. Compare the nodal values of u_h with the finite-difference solution for the same h (lumped load versus consistent load).

19.4 Triangular P_1 elements in two dimensions

Let $\Omega \subset \mathbb{R}^2$ be a polygon and $\mathcal{T}_h$ a triangulation into triangles K of diameter at most h, with no hanging nodes. The space

$$V_h = \left\{ v_h \in C(\overline{\Omega}) : v_h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h,\ v_h|_{\partial\Omega} = 0 \right\}$$

is a subspace of $H_0^1(\Omega)$. Degrees of freedom are the values at interior vertices. On each triangle the three barycentric coordinates are the local hats; the element stiffness

$$A_{ij}^K = \int_K \nabla\lambda_i \cdot \nabla\lambda_j \, dx$$

is computed from the vertex coordinates, then scattered into the global sparse matrix. Shape-regularity (a uniform bound on h_K/ρ_K, with ρ_K the inradius) yields the same rates as in one dimension:

$$\|u - u_h\|_{H^1(\Omega)} \le Ch|u|_{H^2}, \qquad \|u - u_h\|_{L^2(\Omega)} \le Ch^2|u|_{H^2},$$

provided $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Higher-order elements P_k replace the h powers by h^k in H^1 and h^{k+1} in L^2, again by Céa plus interpolation. Mixed or discontinuous spaces are used for saddle-point problems (Stokes) and for $H(\mathrm{div})$ fluxes; they lie outside the present notes.

19.5 A remark on the heat equation

The same space V_h applies to $\partial_t u = \Delta u$ with Dirichlet data. Backward Euler in time and Galerkin in space read

$$\Big(\frac{u_h^{n+1} - u_h^n}{\tau}, v_h\Big) + a(u_h^{n+1}, v_h) = 0 \qquad \forall v_h \in V_h,$$

i.e. $(M + \tau A)U^{n+1} = MU^n$ with mass matrix $M_{ij} = (\varphi_j, \varphi_i)$ and stiffness A as above. The scheme is unconditionally stable in L^2 because A is SPD. For smooth solutions the error is $O(\tau + h^2)$ in L^2 and $O(\tau + h)$ in H^1. Forward Euler requires a CFL restriction of the form $\tau \le Ch^2$, as in the finite-difference case, now coming from the generalized eigenvalue bound of $M^{-1}A$.

19.6 FD, FEM, and spectral methods

	finite differences	finite elements	Fourier spectral
unknowns	grid values	coefficients in V_h	Fourier modes
natural setting	rectangles	polygons / weak form	periodic torus
stability	discrete max. principle	coercivity on V_h	modal decay
smooth data	algebraic $O(h^2)$	algebraic $O(h^k)$	spectral
Neumann data	ghost points	boundary integral	cosine / even extension

Finite differences remain the most direct scheme on a box. Finite elements are the natural discretization of the weak theory already used for existence. Fourier methods, developed in the next part, exploit a global eigenbasis when the domain and the coefficients allow it.

Part V
Fourier spectral methods

20 Fourier spectral methods and foundations

20.1 Scope and a model problem

Finite-difference stencils from Parts I–II are local and typically of fixed algebraic order. Galerkin finite elements (Part IV) discretize the same weak form on a mesh. Fourier spectral methods represent the unknown in a global trigonometric basis, convert constant-coefficient differential operators into multiplication in frequency space, and, for smooth periodic data, converge faster than any polynomial rate.

Model problem: convection–diffusion

Convection–diffusion equation

$$\partial_t u + \mathbf{V} \cdot \nabla u = \nu \Delta u + f, \qquad \nu > 0, \tag{42}$$

where $\nabla = (\partial_x, \partial_y, \partial_z, \dots)$, $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz} + \dots$, $u(t, \mathbf{x}) \in \mathbb{R}$. Typical applications include heat conduction, pollutant diffusion, probability densities in statistical mechanics, and option pricing in finance.

Remark 20.1 (Convergence needs analytical information)**.** For the heat equation (parabolic)

$$\begin{cases} \partial_t u - \Delta u = 0 \qquad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \end{cases}$$

time discretization (implicit Euler)

$$\frac{u^{n+1}-u^n}{\tau} - \Delta u^{n+1} = 0, \qquad \tau = \text{time-step size}.$$

On $[t_n, t_{n+1}]$, $t_n = n\tau$, if $u(t_n) = u^n$, integrating yields

$$u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} \Delta u(s)\, ds.$$

Hence

$$\begin{aligned} u(t_{n+1}) &= u^n + \int_{t_n}^{t_{n+1}} \Delta u^{n+1}\, ds + \int_{t_n}^{t_{n+1}} \big(\Delta u(s) - \Delta u^{n+1}\big)\, ds \\ &= u^{n+1} + \int_{t_n}^{t_{n+1}} \big(\Delta u(s) - \Delta u^{n+1}\big)\, ds, \end{aligned}$$

so the local error

$$u(t_{n+1}) - u^{n+1} = \int_{t_n}^{t_{n+1}} \big(\Delta u(s) - \Delta u^{n+1}\big)\, ds$$

depends on Δu of the analytic solution u (regularity).

20.2 Hilbert space setup for Fourier spectral methods

Definition 20.2 (Inner product / Hilbert space)**.** A map $(\,\cdot\,,\,\cdot\,) : H \times H \to \mathbb{R}$ is an inner product if

(i) $(u, v) = (v, u)$ for all $u, v \in H$;

(ii) $u \mapsto (u, v)$ is linear for each fixed v;

(iii) $(u, u) \geq 0$ for all $u \in H$;

(iv) $(u, u) = 0$ iff $u = 0$.

Define $\|u\| := \sqrt{(u, u)}$. A complete inner-product space is a *Hilbert space*.

Lemma 20.3 (Cauchy–Schwarz)**.** $|(u, v)| \leq \|u\|\,\|v\|$.

Proof. For $\varepsilon > 0$,

$$0 \leq \|u \pm \varepsilon v\|^2 = \|u\|^2 \pm 2\varepsilon(u, v) + \varepsilon^2\|v\|^2,$$

hence

$$\pm(u, v) \leq \frac{1}{2\varepsilon}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2.$$

Take $\varepsilon = \|u\|/\|v\|$ (the case $\|u\| = 0$ or $\|v\| = 0$ is trivial). □

Example 20.4 ($L^2(\Omega)$)**.** $(u, v) := \int_\Omega uv\, dx$ makes $L^2(\Omega)$ a Hilbert space.

Definition 20.5 (Orthogonality / ONB)**.** (i) $u, v \in H$ are orthogonal if $(u, v) = 0$.

(ii) $\{w_k\}_{k=1}^\infty \subset H$ is orthonormal if $(w_k, w_l) = \delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$.

(iii) If $\{w_k\}$ is an orthonormal basis (ONB) and $u \in H$, then

$$u = \sum_{k=1}^\infty (u, w_k)\, w_k, \qquad \|u\|^2 = \sum_{k=1}^\infty (u, w_k)^2.$$

ONB for $L^2([-\pi,\pi])$

Exercise 20.6. Find an ONB for $L^2([-\pi,\pi])$. (The answer is not unique!)

- **Sol 1.** $\{1, x, x^2, \dots\}$ is a basis but not orthonormal $\implies$ use Gram–Schmidt.
- **Sol 2.** The trigonometric system

$$I = \left\{ \tfrac{1}{\sqrt{2\pi}},\ \tfrac{\sin x}{\sqrt{\pi}},\ \tfrac{\sin 2x}{\sqrt{\pi}},\ \dots,\ \tfrac{\cos x}{\sqrt{\pi}},\ \tfrac{\cos 2x}{\sqrt{\pi}},\ \dots \right\}$$

  is orthonormal. For $n \neq 0$,

$$\int_{-\pi}^{\pi} \frac{\sin^2 nx}{\pi}\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2}\, dx = 1,$$

  and for $n \neq m$,

$$\int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \int_{-\pi}^{\pi} \big(\cos((n-m)x) - \cos((n+m)x)\big)\, dx = 0.$$

  Similarly for cosines, so I is orthonormal.

Remark 20.7. $\dim L^2(-\pi,\pi) = \infty$. Completeness of I as an ONB is the classical Fourier completeness theorem.

ONB for $L^2(\mathbb{R})$?

Polynomials $1, x, \dots$ and $\sin nx, \cos nx$ are *not* in $L^2(\mathbb{R})$, so the previous constructions fail. One idea is to localize Fourier modes, e.g. $\chi_{[n,n+1]}(x)\, e^{2\pi i k(x-n)}$ (and analogues).

Linear operators and matrices

Definition 20.8. A map $A : X \to Y$ between Banach spaces is a *linear operator* if

$$A[\lambda u + \mu v] = \lambda A u + \mu A v \qquad \forall u, v \in X,\ \lambda, \mu \in \mathbb{R}.$$

Range $R(A) := \{Au : u \in X\}$, null space $N(A) := \{u : Au = 0\}$. A is bounded if

$$\|A\| := \sup_{u\neq 0} \frac{\|Au\|}{\|u\|} = \sup_{\|u\|=1} \|Au\| < \infty.$$

Remark 20.9. If X, Y are finite-dimensional, $X \simeq \mathbb{R}^n$, $Y \simeq \mathbb{R}^m$, then $A \in M_{m\times n}(\mathbb{R})$: discretization turns PDE operators into matrices.

Definition 20.10 (Adjoint / symmetric)**.** If $A : H \to H$ is bounded linear, its adjoint A^* satisfies $(Au, v) = (u, A^* v)$ for all $u, v \in H$. A is symmetric if $A^* = A$.

Example 20.11. $\frac{d}{dx}$ on $C^\infty([a,b])$ is linear but *not bounded*: $\frac{d}{dx}\sin nx = n\cos nx$, so $\left\|\frac{d}{dx}\sin nx\right\|_C = n \to \infty$ as $n \to \infty$.

20.3 Fourier series on the torus

On $\mathbb{T} = [0, 2\pi]$ (periodic identification of the endpoints) write

$$u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, e^{ikx}, \qquad \hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)\, e^{-ikx}\, dx. \tag{43}$$

The complex basis $\{e^{ikx}\}_{k\in\mathbb{Z}}$ is orthogonal in $L^2(\mathbb{T})$ and equivalent to the real trigonometric ONB of the previous subsection. Parseval's identity reads

$$\frac{1}{2\pi}\int_0^{2\pi} |u(x)|^2\, dx = \sum_{k\in\mathbb{Z}} |\hat{u}_k|^2.$$

Proposition 20.12 (Differentiation and convolution)**.** *If u is 2π-periodic and C^1, then $\widehat{u'}(k) = ik\,\hat{u}_k$. More generally, $\widehat{\partial_x^m u}(k) = (ik)^m \hat{u}_k$. If $u, v \in L^1(\mathbb{T})$ and $(u * v)(x) = \frac{1}{2\pi}\int_0^{2\pi} u(x-y)v(y)\,dy$, then $\widehat{u * v}(k) = \hat{u}_k \hat{v}_k$.*

Thus $-\partial_{xx}$ becomes multiplication by k^2 on each mode: the heat equation $\partial_t u = \partial_{xx} u$ on the circle reduces to the ODE $\partial_t \hat{u}_k = -k^2 \hat{u}_k$.

Theorem 20.13 (Decay of coefficients)**.** *If $u \in C^p(\mathbb{T})$, then $\hat{u}_k = o(|k|^{-p})$ as $|k| \to \infty$ (integrate by parts p times). If u is analytic, the coefficients decay exponentially: $|\hat{u}_k| \le Ce^{-\alpha|k|}$ for some $\alpha > 0$.*

Truncating (43) at $|k| \le N$ therefore yields spectral accuracy for smooth periodic data: the error is smaller than any negative power of N, and exponentially small if u is analytic.

20.4 The Fourier transform on $\mathbb{R}^n$

On the whole space the analogous tool is the Fourier transform

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x)\, e^{-ix\cdot\xi}\, dx, \qquad u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi)\, e^{ix\cdot\xi}\, d\xi, \tag{44}$$

initially for $u \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz class) and then by density on L^1, L^2, and tempered distributions.

Theorem 20.14 (Basic calculus of the Fourier transform)**.** *(i)* ***Inversion.*** *If $u, \hat{u} \in L^1(\mathbb{R}^n)$, then the inversion formula in (44) holds pointwise after a continuous representative is chosen.*

(ii) ***Plancherel.*** *The map $u \mapsto \hat{u}$ extends to a scaled isometry of L^2:*

$$\int_{\mathbb{R}^n} |u(x)|^2\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2\, d\xi.$$

(iii) ***Derivatives.*** *$\widehat{\partial_{x_j} u}(\xi) = i\xi_j \hat{u}(\xi)$ and $\widehat{\Delta u}(\xi) = -|\xi|^2 \hat{u}(\xi)$.*

(iv) ***Convolution.*** *$\widehat{u * v} = \hat{u}\,\hat{v}$, where $(u * v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)\,dy$.*

(v) ***Riemann–Lebesgue.*** *If $u \in L^1(\mathbb{R}^n)$, then $\hat{u}(\xi) \to 0$ as $|\xi| \to \infty$.*

Example 20.15 (Heat kernel)**.** The Gaussian $G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ $(t > 0)$ has Fourier transform $\widehat{G_t}(\xi) = e^{-t|\xi|^2}$. Consequently the whole-space heat equation $\partial_t u = \Delta u$ is solved by

$$\hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{u}_0(\xi), \qquad u(t) = G_t * u_0,$$

recovering the same kernel already used in Part II.

Example 20.16 (Poisson equation on $\mathbb{R}^n$)**.** For $-\Delta u = f$ with u, f decaying at infinity, $\widehat{u}(\xi) = |\xi|^{-2} \hat{f}(\xi)$ for $\xi \ne 0$. The multiplier $|\xi|^{-2}$ is locally integrable if $n \ge 3$, consistent with the Newtonian potential of Part I.

Remark 20.17 (Sobolev norms via Fourier)**.** For $s \in \mathbb{R}$,

$$\|u\|_{H^s(\mathbb{R}^n)}^2 := (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2\, d\xi$$

makes the Fourier transform a convenient definition of fractional Sobolev spaces, to be compared with the weak-derivative approach of Part III.

20.5 Discrete Fourier transform and spectral collocation

On the uniform grid $x_j = 2\pi j/N$, $j = 0, \ldots, N-1$, the discrete Fourier transform (DFT)

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j)\, e^{-ikx_j}, \qquad k = -\tfrac{N}{2}, \ldots, \tfrac{N}{2} - 1$$

is inverted by

$$u(x_j) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k\, e^{ikx_j}.$$

The FFT evaluates both maps in $O(N \log N)$ operations. Spectral collocation replaces ∂_x by multiplication by ik in this basis (and Δ by $-|k|^2$), then returns to physical space by the inverse FFT. For a linear constant-coefficient PDE on the torus the resulting ODE system in the coefficients $\hat{u}_k(t)$ is diagonal.

Exercise 20.18. Using (44), solve $\partial_t u = \Delta u$ on $\mathbb{R}$ with $u(0,x) = e^{-x^2}$ by transforming, multiplying by $e^{-t\xi^2}$, and inverting.

20.6 Weak convergence (Fourier-relevant examples)

Definition 20.19 (Weak convergence)**.** A sequence $\{u_k\} \subset X$ converges weakly to u, written $u_k \rightharpoonup u$, if

$$\langle u^*, u_k \rangle \to \langle u^*, u \rangle \qquad \forall u^* \in X^*.$$

Example 20.20. In $L^p(\Omega)$, $f_k \rightharpoonup f$ means

$$\int_\Omega g f_k\, dx \to \int_\Omega g f\, dx \qquad \forall g \in L^{p'}(\Omega),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. ($L^p$ is reflexive for $1 < p < \infty$.)

Example 20.21 ($\sin kx \rightharpoonup 0$)**.** Let $f_k = \sin kx$ in $L^2([-\pi,\pi])$. Then $\sin kx \rightharpoonup 0$ as $k \to \infty$. Indeed, for all $f \in L^2([-\pi,\pi])$ one needs $\int_{-\pi}^{\pi} \sin kx \cdot f\, dx \to 0$. By density assume $f \in C_c^\infty([-\pi,\pi])$; integration by parts gives

$$\begin{aligned} \int_{-\pi}^{\pi} \sin kx \cdot f\, dx &= -\frac{1}{k}\Big(f \cos kx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos kx\, f'(x)\, dx \Big) \\ &= \frac{1}{k} \int_{-\pi}^{\pi} \cos kx\, f'(x)\, dx, \end{aligned}$$

and $\big|\int \cos kx\, f'\big| \le 2\pi \|f\|_{C^1}$, so the integral tends to 0 as $k \to \infty$.

Remark 20.22 (Weak $\neq$ strong)**.**

(i) **Weak but not strong.** Let $f_k = \chi_{[k,k+1]}$ in $L^2([0,\infty))$. For any $g \in C_c^\infty([0,\infty))$ with $\operatorname{supp} g \subseteq [0,M]$, if $k \ge M$ then $\int_0^\infty f_k g = \int_k^{k+1} g = 0$, so $f_k \rightharpoonup 0$. But

$$\|f_k\|_{L^2([0,\infty))}^2 = \int_k^{k+1} 1\, dx = 1 \neq 0,$$

so $\{f_k\}$ does not converge strongly to 0.

(ii) **Strong $\Rightarrow$ weak.** If $\|f_k - f\| \to 0$, then by Cauchy–Schwarz

$$\big|\langle u^*, f_k - f \rangle\big| \le \|u^*\|\, \|f_k - f\| \to 0.$$

21 Fourier expansions for IBVP and spectral time stepping

21.1 Fourier sine / cosine expansions for IBVP

21.1.1 Separation of variables (Dirichlet)

For

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, L), \\ u(0, x) = f(x), \\ u(t, 0) = u(t, L) = 0, \end{cases}$$

set $u = X(x)T(t)$. Then $X''/X = \dot{T}/T = C$. Only $C = -\lambda^2 < 0$ yields physically decaying modes:

$$u = (A \cos \lambda x + B \sin \lambda x)\, e^{-\lambda^2 t}.$$

Dirichlet forces $A = 0$ and $\lambda = n\pi/L$, hence

$$u(t, x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi}{L} x\Big) e^{-(n\pi/L)^2 t}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi}{L} x\Big)\, dx.$$

21.1.2 Non-homogeneous Neumann: particular solution + cosine series

Example 21.1.

$$\begin{cases} \partial_t u = \partial_{xx} u, & 0 < x < 1,\ t > 0, \\ u_x(t, 0) = 0, \quad u_x(t, 1) = 1, \\ u(0, x) = -\dfrac{2}{\pi} \cos\big(\dfrac{\pi}{2} x\big). \end{cases}$$

Try $w(t, x) = ax^2 + bx + Ct$. Matching the PDE gives $C = 2a$; matching Neumann data gives $a = \frac{1}{2}$, $b = 0$, so $w = \frac{1}{2}x^2 + t$. Set $u = w + v$:

$$\begin{cases} \partial_t v = \partial_{xx} v, \\ \partial_x v(t, 0) = \partial_x v(t, 1) = 0, \\ v(0, x) = -\dfrac{2}{\pi} \cos\big(\dfrac{\pi}{2} x\big) - \dfrac{x^2}{2} =: f(x). \end{cases}$$

Separation with homogeneous Neumann yields the cosine expansion

$$v = b_0 + \sum_{n=1}^{\infty} b_n \cos\Big(\frac{n\pi x}{L}\Big) e^{-(n\pi/L)^2 t},$$

with $(L = 1)$

$$b_0 = \int_0^1 f(x)\, dx, \qquad b_n = 2 \int_0^1 f(x) \cos(n\pi x)\, dx \quad (n \geq 1).$$

Remark 21.2 (Summary: BC $\leftrightarrow$ series)**.**
- Dirichlet $= 0$: sine series; $\neq 0$: particular solution + sine.
- Neumann $= 0$: cosine series; $\neq 0$: particular solution + cosine.
- Odd extension of a Dirichlet datum on $[0, L]$ produces a $2L$-periodic function; even extension does the same for Neumann (even symmetry).

21.2 Fourier spectral method on a periodic domain

21.2.1 Periodic heat equation

$$\begin{cases} \partial_t u = \partial_{xx} u & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \quad u^{(n)}(0) = u^{(n)}(2\pi) & \text{(periodic BC)}, \\ u(0, x) = u_0(x). \end{cases}$$

Separation with periodic BC forces eigenvalues $\lambda_n = n \in \mathbb{Z}$,

$$u_n(t, x) = e^{-n^2 t}\big(A_n \cos nx + B_n \sin nx\big).$$

Matching u_0 gives

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} u_0\,dx, \qquad A_n = \frac{1}{\pi}\int_0^{2\pi} u_0 \cos nx\,dx, \qquad B_n = \frac{1}{\pi}\int_0^{2\pi} u_0 \sin nx\,dx.$$

21.2.2 Time stepping on the Fourier side

On modes $\hat{u}(k)$, ∂_{xx} becomes multiplication by $-|k|^2$.

- **Backward Euler:**
$$\hat{u}^{m+1}(k) = \frac{\hat{u}^m(k)}{1 + |k|^2\Delta t}.$$
Since $1 + |k|^2\Delta t > 1$, the scheme is unconditionally stable; easy to compute; best suited to periodic (or simple Dirichlet/Neumann) BC.

- **Forward Euler:**
$$\hat{u}^{m+1}(k) = (1 - \Delta t|k|^2)\,\hat{u}^m(k),$$
requiring $\Delta t \lesssim 2/|k|^2_{\max}$ (stability bad). Spatial truncation in N modes is spectrally accurate.

- **Crank–Nicolson:**
$$\hat{u}^{m+1}(k) = \frac{1 - \frac{1}{2}\Delta t|k|^2}{1 + \frac{1}{2}\Delta t|k|^2}\,\hat{u}^m(k),$$
with amplification factor magnitude < 1 (unconditional stability).

Remark 21.3 (Accuracy checks / spectral BC)**.** Practical accuracy checks: (i) compare against a reference with very small Δt; (ii) method of manufactured ("forced") solutions. Spectral methods extend to simple Dirichlet / Neumann conditions via odd/even extensions or particular solutions as above.

Example 21.4 (Spectral solution of a periodic heat equation)**.** Take $u_0(x) = \cos x$ on $[0, 2\pi]$. Then $\hat{u}_{\pm 1}(0) = \frac{1}{2}$ and all other coefficients vanish, so

$$u(t, x) = e^{-t}\cos x$$

is both the exact solution and the one-mode spectral approximation: there is no spatial truncation error. Forward Euler on this mode is $\hat{u}^{m+1} = (1 - \Delta t)\hat{u}^m$, which is stable for $\Delta t \le 2$. Backward Euler, $\hat{u}^{m+1} = \hat{u}^m/(1 + \Delta t)$, is stable for every $\Delta t > 0$ and reproduces the exact damping $e^{-\Delta t}$ up to $O((\Delta t)^2)$ per step.

Exercise 21.5. For $u_0(x) = \cos(8x)$ on $[0, 2\pi]$, write the forward Euler restriction on Δt coming from this single mode, and compare it with the restriction coming from a truncation $|k| \le N = 64$.

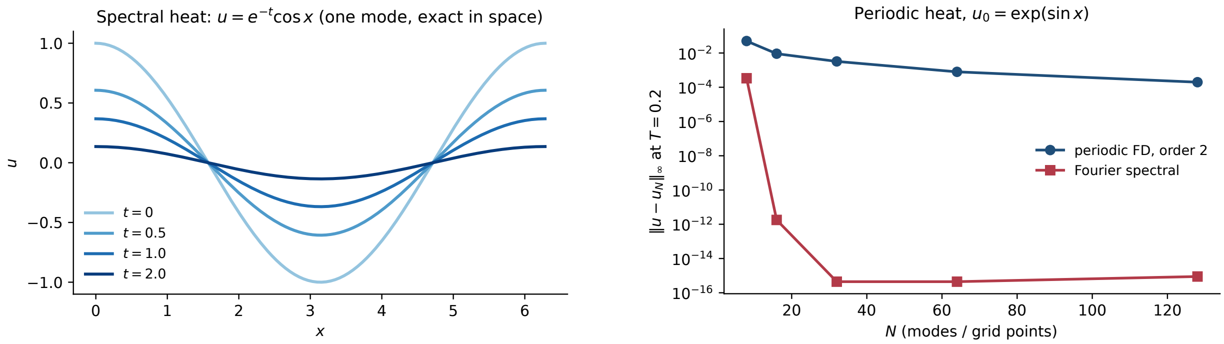


Figure 8: Left: one-mode spectral solution $u = e^{-t}\cos x$. Right: periodic heat equation with $u_0 = \exp(\sin x)$; Fourier collocation reaches roundoff by $N \approx 32$, while second-order finite differences decay only algebraically.

Part VI
Phase-field models and fluids

22 Phase-field models, exponential integrators, and fluids

22.1 Phase-field models

Classical phase-field models:

- Allen–Cahn (1979): Fe–Mn mixing / interface motion;
- Cahn–Hilliard (1958): solid–liquid interfaces;
- MBE: molecular beam epitaxy (thin-film growth).

Applications: dendritic growth, lithium batteries, crystal precipitation.

On the periodic domain $\mathbb{T}^d$ (repeating microstructure; convenient for spectral methods), following the Li–Tang–Qiao (*SINUM* 2016) framework, the model equations are

$$\partial_t u = \nu\Delta u + u - u^3 \qquad \text{(AC)}, \tag{45}$$
$$\partial_t u = -\Delta\big(\nu\Delta u + u - u^3\big) \qquad \text{(CH)}. \tag{46}$$

Three leading journals in numerical analysis: *SINUM*, *Math. Comp.*, *Numer. Math.*

Remark 22.1 (AC maximum principle)**.** For smooth solutions,

$$\|u(t)\|_\infty \le \max\big\{\|u_0\|_\infty,\, 1\big\} \qquad \forall\, 0 \le t \le T.$$

The standard proof takes $g = u^2$, $g^\varepsilon = g - \varepsilon t$, and at a maximum point uses the AC equation together with $u\Delta u \le 0$ to reach a contradiction if $\|u\|_\infty > 1$; then let $\varepsilon \to 0$.

The 1D stationary equation $\partial_{xx}u + u - u^3 = 0$ has the solution $u = \tanh(x/\sqrt{2})$; the interface-type approximation $\tilde{u} = \tanh(x/\varepsilon)$ has transition-layer width ε.

22.2 Gradient flow and energy dissipation

Define the energy

$$E(u) := \int_{\mathbb{T}^d} \Big(\frac{\nu}{2}|\nabla u|^2 + \frac{(u^2-1)^2}{4}\Big)\, dx$$

(kinetic / potential). For AC,

$$\frac{dE}{dt} = \int \nu \nabla u \cdot \nabla u_t + (u^3 - u)u_t \, dx = \int u_t(-\nu\Delta u + u^3 - u)\, dx = -\|u_t\|_2^2 \le 0.$$

For CH (mass conservation $\bar{u} = 0$ is needed to define $(-\Delta)^{-1}$),

$$\frac{dE}{dt} = -\big\||\nabla|^{-1} u_t\big\|_2^2 \le 0.$$

Gradient flow $\Rightarrow$ H^1 a-priori control.

Scaling-critical space (AC)

For $\partial_t u = \Delta u + u - u^3$, the scaling $\lambda^s u(\lambda^2 t, \lambda x)$ yields the critical index $s = 1$, hence

$$\|\lambda u(\lambda^2 t, \lambda x)\|_{\dot{H}^k(\mathbb{R}^d)} = \lambda^{1+k-d/2} \|u\|_{\dot{H}^k},$$

and the critical index satisfies $k + 1 = d/2$ (e.g. $d = 2$: L^2; $d = 3$: $\dot{H}^{1/2}$). Thus H^1 a-priori control $\Rightarrow$ GWP.

Interface motion

- AC: $\{u = 0\}$ moves by mean curvature; in 2D this is curve shortening $V = -K$; in 3D it is mean curvature flow (non-conservative). References: Alikakos–Bates–Chen (ABC), Pego (89), Chen (94), Fei Lin, Wang–Zhang (23), etc.
- CH: Mullins–Sekerka flow (mass-conserving).

22.3 Numerical illustrations

A Fourier IMEX discretization of (45)–(46) on $\mathbb{T}$ and $\mathbb{T}^2$ produces the dynamics below.

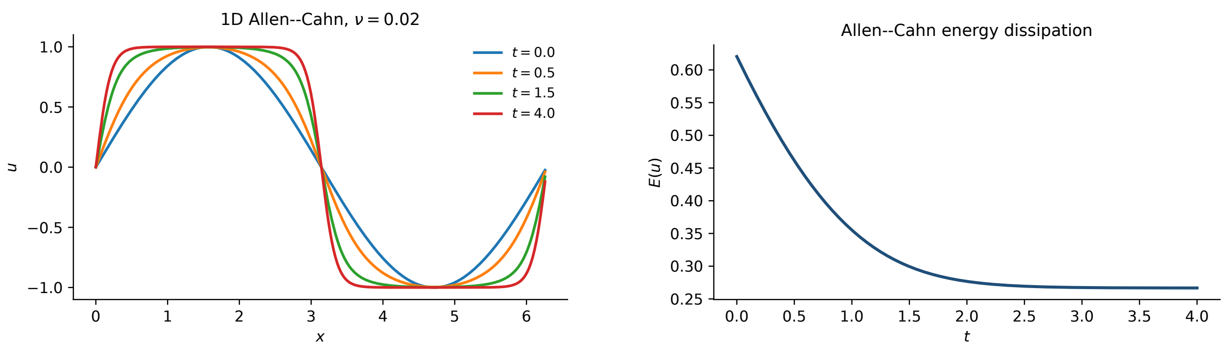


Figure 9: 1D Allen–Cahn from $u_0 = \sin x$. The solution relaxes toward a heteroclinic profile connecting ± 1, and the Ginzburg–Landau energy decreases.

Further seven-bubble, LEI, sine–Gordon, MBE, SQG and FBSDE experiments are collected in Section 24.

22.4 Energy-stable spectral schemes for CH

For the evolution equation $\partial_t u = Gu + f(u)$, the naive semi-implicit scheme

$$\frac{u^{n+1} - u^n}{\tau} = Gu^{n+1} + f(u^n), \qquad G \in \{\Delta,\, -\Delta^2,\, -(-\Delta)^{\alpha+1}\}\ (\alpha \in (0,1)),$$

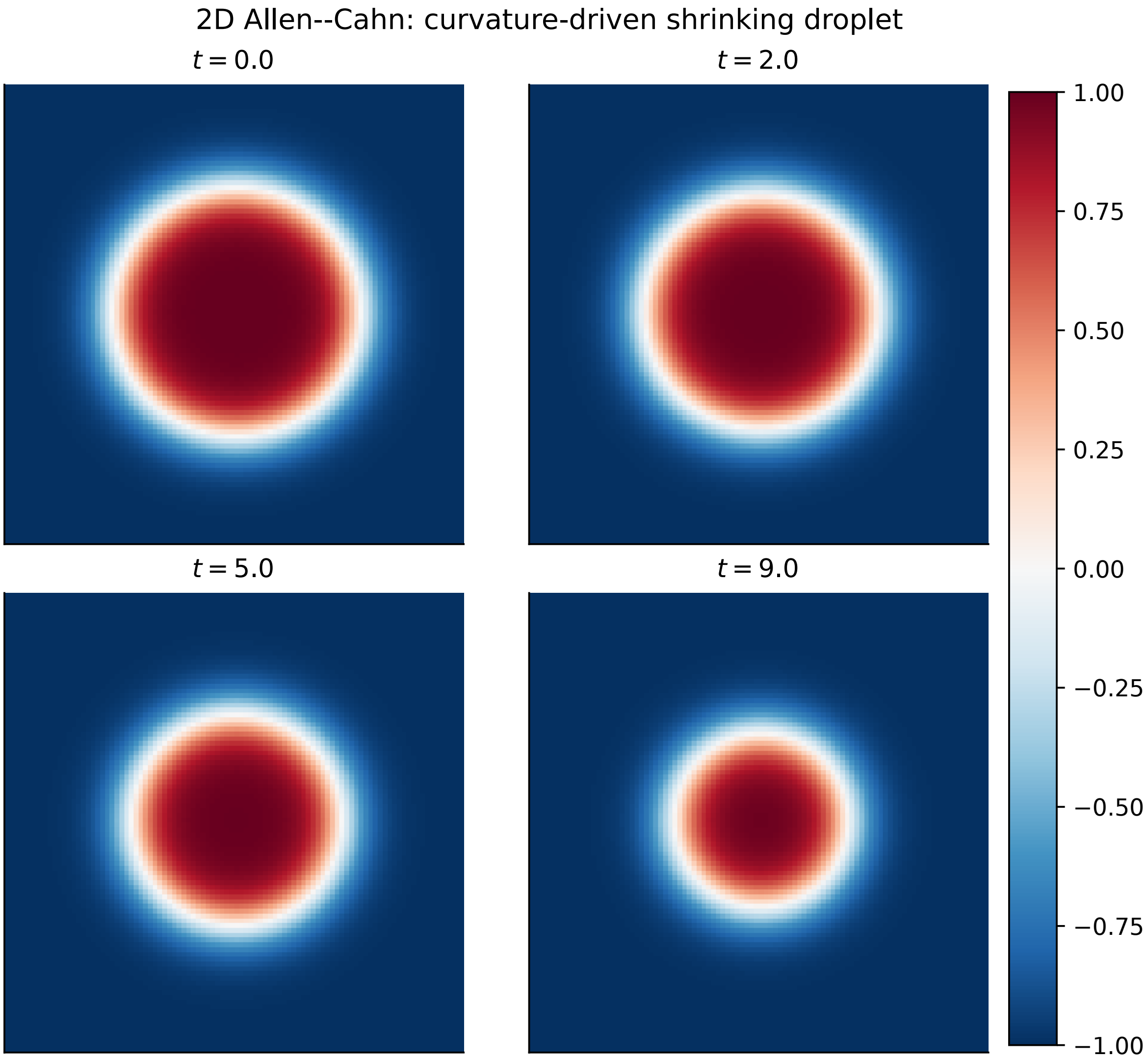


Figure 10: 2D Allen–Cahn: a circular droplet shrinks by mean curvature.

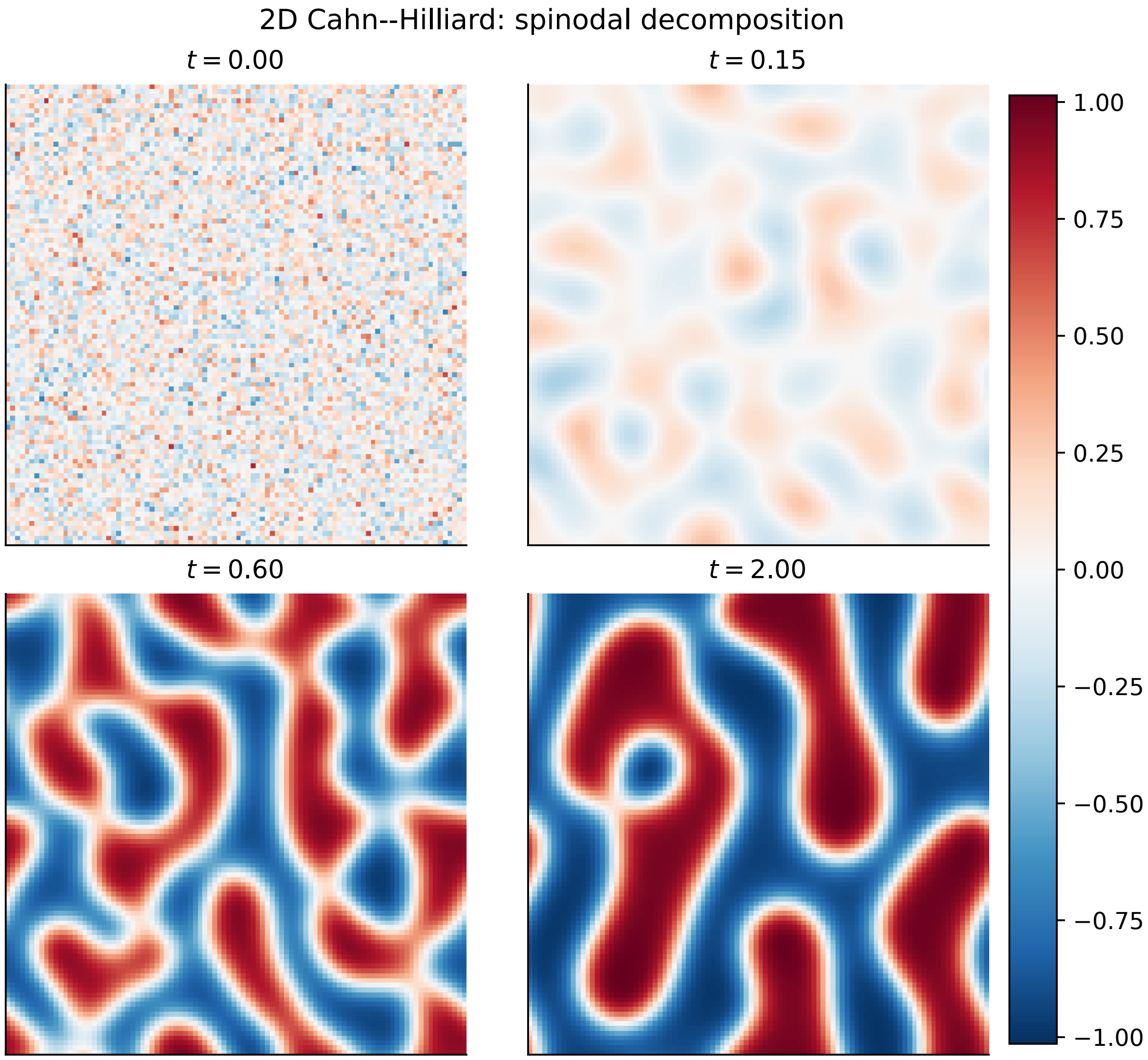


Figure 11: 2D Cahn–Hilliard spinodal decomposition from small random data, computed with the stabilized spectral scheme (47).

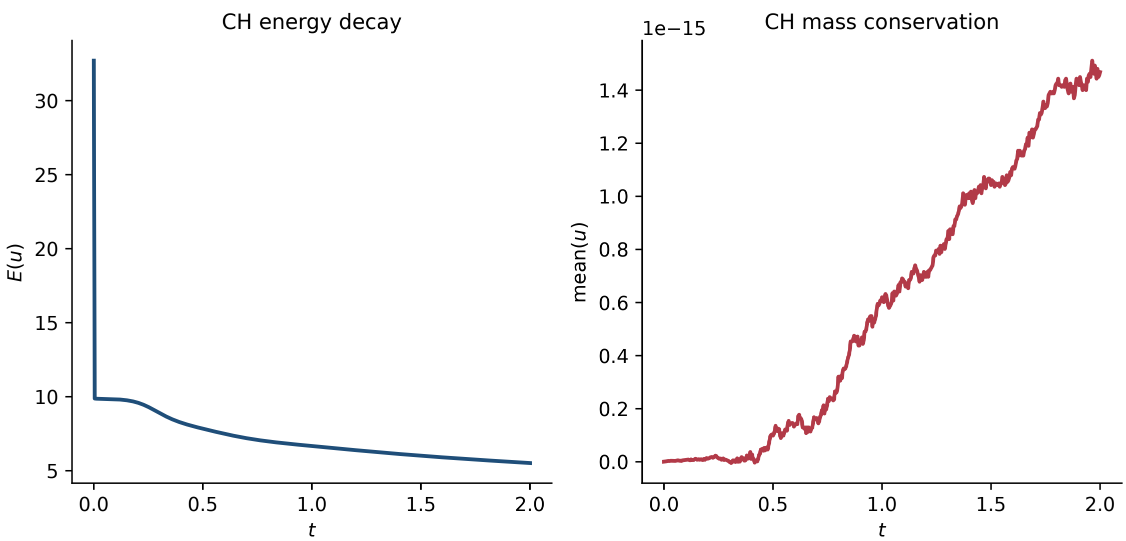


Figure 12: Cahn–Hilliard energy decay and mass conservation (the mean is at roundoff).

i.e. $u^{n+1} = (u^n + \tau f(u^n))/(I - G\tau)$. Below we mainly discuss $-\Delta^2$-type CH (Li–Qiao–Tang 2016): the goal is energy stability $\tilde{E}(u^{n+1}) \le \tilde{E}(u^n)$, unconditionally for any $\tau > 0$, which requires a stabilization term; $f(u) = u^3 - u$.

Spectral method (frequency truncation $P_N = P_{\le N}$) with stabilization ($\star$):

$$\frac{u^{n+1} - u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1} - u^n) + \Delta P_N f(u^n), \qquad u^0 = P_N u_0. \tag{47}$$

Theorem 22.2 (Li–Qiao–Tang; energy decay)**.** *If $A \gg \nu^{-1}|\log\nu|^2$ (note: not sharp), then $E(u^{n+1}) \le E(u^n)$.*

Sketch. Take the inner product of ($\star$) against $\langle\,\cdot\,, (-\Delta)^{-1}(u^{n+1} - u^n)\rangle$, yielding

$$\begin{aligned}\frac{1}{\tau}\big\||\nabla|^{-1}(u^{n+1} - u^n)\big\|_2^2 = \frac{\nu}{2}\Big(\|\nabla u^n\|_2^2 - \|\nabla u^{n+1}\|_2^2 - \|\nabla(u^{n+1} - u^n)\|_2^2\Big)\\ - A\|u^{n+1} - u^n\|_2^2 - (P_{\le N} f(u^n), u^{n+1} - u^n).\end{aligned}$$

Rearranging and using the integral remainder for $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$ ($F' = f$),

$$F(u^{n+1}) - F(u^n) = f(u^n)(u^{n+1} - u^n) + \int_{u^n}^{u^{n+1}} (3s^2 - 1)(u^{n+1} - s)\, ds,$$

one obtains

$$\begin{aligned}&E(u^{n+1}) - E(u^n) + \frac{1}{\tau}\big\||\nabla|^{-1}(u^{n+1} - u^n)\big\|_2^2 + \frac{\nu}{2}\|\nabla(u^{n+1} - u^n)\|_2^2\\ &\qquad + \Big(A + \tfrac{1}{2}\Big)\|u^{n+1} - u^n\|_2^2\\ &\le \|u^{n+1} - u^n\|_2^2\Big(\|u^n\|_\infty^2 + \tfrac{1}{2}\|u^{n+1}\|_\infty^2\Big).\end{aligned}$$

By interpolation $\frac{1}{\tau}\big\||\nabla|^{-1}\delta u\big\|_2^2 + \frac{\nu}{2}\|\nabla\delta u\|_2^2 \ge \sqrt{2\nu/\tau}\,\|\delta u\|_2^2$, so whenever

$$A + \tfrac{1}{2} + \sqrt{\frac{2\nu}{\tau}} \ge \|u^n\|_\infty^2 + \tfrac{1}{2}\|u^{n+1}\|_\infty^2 \tag{48}$$

one gets energy dissipation. After the inductive hypothesis $E^n \le \cdots \le E_0$, one must control $\|u^n\|_\infty, \|u^{n+1}\|_\infty$ (below we take $\mathbb{T}^2$). □

The Fourier form is convenient for computation:

$$u^{n+1} = \frac{I - A\tau\Delta}{I + \nu\tau\Delta^2 - A\tau\Delta}\, u^n + \frac{\tau\Delta P_N}{I + \nu\tau\Delta^2 - A\tau\Delta}\, f(u^n).$$

22.5 L^∞ bounds via log-Sobolev / Littlewood–Paley

Lemma 22.3 (Brezis–Gallouët type on $\mathbb{T}^2$)**.** *If $\bar{f} = 0$, then for $s > 1$,*

$$\|f\|_\infty \lesssim \|f\|_{\dot{H}^1}\log\big(1 + \|f\|_{\dot{H}^s}\big) + 1.$$

Sketch. Littlewood–Paley: $\|f\|_\infty \lesssim \sum_{j\ge -1}\|\Delta_j f\|_\infty$. Splitting at J and using Bernstein,

$$\|f\|_\infty \lesssim J\|f\|_{\dot{H}^1} + 2^{(1-s)J}\|f\|_{\dot{H}^s}.$$

Optimizing $H(J) = J\|f\|_{\dot{H}^1} + 2^{(1-s)J}\|f\|_{\dot{H}^s}$ yields $J \sim \log(1 + \|f\|_{\dot{H}^s})$. □

In the spectral scheme take $s = 3/2$. Fourier-multiplier estimates give

$$\|u^{n+1}\|_{H^1} \le \|u^n\|_{H^1} + \frac{1}{A}\|(u^3-u)^n\|_{H^1},$$

and

$$\begin{aligned}\|u^{n+1}\|_{\dot H^{3/2}} &\lesssim \Big(\frac{1}{A\tau} + \frac{A}{\nu}\Big)\|u^n\|_2 + \frac{1}{\nu}\big\||\nabla|^{-1/2}((u^n)^3 - u^n)\big\|_2 \\ &\lesssim \Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big)(E^n+1).\end{aligned}$$

Hence

$$\|u^n\|_\infty \le 1 + C_1\sqrt{\frac{2E_0}{\nu}}\log\Big(1 + C_2\Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big)(E_0+1)\Big) \lesssim_{E_0} \nu^{-1/2}\big(1 + \log A + |\log\nu| + |\log\tau|\big) + 1.$$

Let m_0 denote this logarithmic bound; then choosing

$$A \gg_{E_0} m_0^2 \qquad (\text{or } A + \tfrac{1}{A}\sqrt{2E_0/\nu} \gg_{E_0} m_0^2)$$

closes (48).

22.6 L^2 error estimates for the CH scheme

Theorem 22.4 (L^2 error)**.**

$$\|u(m\tau) - u^m\|_2 \le A\, e^{Cm\tau}\, C_2\big(N^{-s} + \tau\big).$$

Tool: discrete Grönwall. If $(y^{n+1} - y^n)/\tau \le \tilde\alpha_n y^n + \tilde\beta_n$, then

$$y^m \le \exp\Big(\tau\sum_{n=0}^{m-1}\tilde\alpha_n\Big)y_0 + \tau\sum_{k=0}^{m-1}\exp\Big(\tau\sum_{j=k+1}^{m-1}\tilde\alpha_j\Big)\tilde\beta_k;$$

in particular

$$y^m \le \exp\Big(\tau\sum\tilde\alpha_n\Big)\Big(y_0 + \tau\sum\tilde\beta_k\Big).$$

Continuous counterpart: $\partial_t y \le \alpha(t)y + \beta(t)$ yields $y(t) \le e^{\int_0^t\alpha}y_0 + \int_0^t e^{\int_s^t\alpha}\beta(s)\,ds$.

Proposition 22.5 (Auxiliary system)**.** *Consider*

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\tau} - A\Delta(u^{n+1}-u^n) = -\nu\Delta^2 u^{n+1} + \Delta P_{\le N} f(u^n),\\ \dfrac{v^{n+1}-v^n}{\tau} - A\Delta(v^{n+1}-v^n) = -\nu\Delta^2 v^{n+1} + \Delta P_{\le N} f(v^n) + \Delta G^n.\end{cases}$$

If $\|v^n\|_\infty + \|\nabla v^n\|_2 + \|\nabla u^n\|_2 \le C$, *then*

$$\begin{aligned}\|u^m - v^m\|_2^2 \le \exp\Big(\frac{C}{\nu}m\tau\Big)\Bigg(&\|u_0 - v_0\|_2^2 + A\tau\|\nabla u_0 - \nabla v_0\|_2^2 \\ &+ \frac{\tau}{\nu}\sum_{n=0}^{m-1}\|G^n\|_2^2\Bigg).\end{aligned}$$

Sketch. Set $e^n = u^n - v^n$. Subtract and take the inner product with e^{n+1}, yielding

$$\begin{aligned}&\frac{1}{2\tau}\big(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 + \|e^{n+1}-e^n\|_2^2\big) \\ &\quad + \frac{A}{2}\big(\|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2 + \|\nabla(e^{n+1}-e^n)\|_2^2\big) + \nu\|\Delta e^{n+1}\|_2^2 \\ &= \big(P_{\le N}(f(u^n) - f(v^n)), \Delta e^{n+1}\big)_{\mathrm{I}} - \big(G^n, \Delta e^{n+1}\big)_{\mathrm{II}}.\end{aligned}$$

For $f(u) = u^3 - u$,

$$f(u^n) - f(v^n) = \int_0^1 f'(v^n + se^n)\, ds \cdot e^n = a_1 e^n + a_2 (v^n)^2 e^n + a_3 v^n (e^n)^2 + a_4 (e^n)^3$$

(write I_1, I_2, I_3 for the last three terms). Young / Sobolev:

$$\begin{aligned} I_1 &\le \tfrac{\nu}{8}\|\nabla e^{n+1}\|_2^2 + \tfrac{C}{\nu}\|e^n\|_2^2, \\ I_2 &\le \|v^n\|_\infty \|e^n\|_4^2 \|\nabla e^{n+1}\|_2 \le \tfrac{C}{\nu}\|e^n\|_2^2 + \tfrac{\nu}{8}\|\nabla e^{n+1}\|_2^2, \\ I_3 &\le \|e^n\|_6^3 \|\nabla e^{n+1}\|_2 \le \tfrac{C}{\nu}\|e^n\|_2^2 + \tfrac{\nu}{8}\|\nabla e^{n+1}\|_2^2, \end{aligned}$$

and $\mathrm{II} \le \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2 + \frac{C}{\nu}\|G^n\|_2^2$. Set $y^n := \|e^n\|_2^2 + A\tau\|\nabla e^n\|_2^2$; then $(y^{n+1} - y^n)/\tau \le (C/\nu) y^n + (C/\nu)\|G^n\|_2^2$, and discrete Grönwall concludes the proof. □

Real case / truncation

Substitute the exact solution into the scheme:

$$\begin{aligned} \frac{u((n+1)\tau) - u(n\tau)}{\tau} &= -\nu\Delta^2 u((n+1)\tau) + A\Delta\big(u((n+1)\tau) - u(n\tau)\big) \\ &\quad + \Delta P_{\le N} f(u(n\tau)) + \Delta P_{>N} f(u(n\tau)) + \Delta G^n, \end{aligned}$$

where (schematically)

$$\begin{aligned} G^n = &-\frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t \Delta u \cdot (t_n - t)\, dt + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t f(u) \cdot (t_{n+1} - t)\, dt \\ &- A\int_{t_n}^{t_{n+1}} \partial_t u\, dt. \end{aligned}$$

Estimate $\tau\sum\|G^n\|_2^2 \lesssim \tau^2(1+A)^2\int_0^{t_m}\|\partial_t\Delta u\|_2^2\, dt$, and (assuming $\|f(u(t))\|_{H^s} \lesssim 1$) $\tau\sum\|P_{>N} f(u(t_n))\|_2^2 \lesssim m\tau N^{-2s}$. Initial projection: $\|u_0 - P_N u_0\|_2 \lesssim N^{-s}$, $\|\nabla u_0 - \nabla P_N u_0\|_2 \lesssim N^{-(s-1)}$. Altogether,

$$\|u^m - u(t_m)\|_2 \lesssim (1+A)\, e^{Ct_m}\, (\tau + N^{-s}), \qquad t_m \le T.$$

Remark 22.6. The exponential contains $e^{Ct_m/\nu}$; usable for fixed ν, but the estimate fails as $\nu \to 0$.

Smoothing for fourth-order heat kernel

$$\int_0^t \big\|e^{-(t-s)\nu\Delta^2}\Delta f\big\|_2\, ds \lesssim \int_0^t (t-s)^{-(2-s)/4}\|f\|_{H^s}\, ds,$$

since on the Fourier side $e^{-(t-s)\nu|\xi|^4}|\xi|^{2-s}|\xi|^s\hat{f}$, and the scaling $\Delta^2 \sim (t-s)^{-1} \Rightarrow |\xi| \sim (t-s)^{-1/4}$.

22.7 Exponential integrators: motivation, IMEX, and BDF2

Exponential integrator methods. For $\partial_t u = \mathcal{L}u + f(u)$:

1st-order semi-implicit

$$\frac{u^{n+1} - u^n}{\tau} = \mathcal{L}u^{n+1} + f(u^n), \qquad u^n \approx u(t_n),\ t_n = n\tau.$$

Taylor: $u(t_{n+1}) - u(t_n) = \tau\partial_t u(t_n) + \frac{\tau^2}{2}\partial_{tt} u(\varsigma)$, truncation error $O(\tau)$; the RHS cancels $\partial_t u(t_n)$.

2nd-order BDF2

$$\frac{u^{n+1} - \frac{4}{3}u^n + \frac{1}{3}u^{n-1}}{\frac{2}{3}\tau} = \mathcal{L}u^{n+1} + f(u^{n+1}).$$

Expanding at t_{n+1}: the constant and τ^2 terms cancel, leaving $\frac{2}{3}\tau\,\partial_t u$, so after dividing by $\frac{2}{3}\tau$ the truncation error is $O(\tau^2)$ (remainder $\sim \tau^2\partial_{ttt}u$).

Remark 22.7 (Truncation vs. regularity)**.** Higher-order truncation $\Leftrightarrow$ higher regularity of the exact solution: $O(\tau) \Leftrightarrow \partial_{tt}u \sim \Delta^2 u$; $O(\tau^2) \Leftrightarrow \partial_{ttt}u \sim \Delta^3 u$. Question: can one lower the regularity requirement on the exact solution?

22.8 First-order exponential integrator (Allen–Cahn)

Model $\partial_t u = \Delta u + f(u)$ (WLOG $\varepsilon = 1$). Duhamel / heat kernel:

$$u(t) = e^{t\Delta}u(t_0) + \int_{t_0}^{t} e^{(t-s)\Delta} f(u(s))\,ds,$$

where $e^{t\Delta} = I + t\Delta + \frac{1}{2!}(t\Delta)^2 + \cdots$, and $\widehat{e^{t\Delta}g}(k) = e^{-t|k|^2}\hat{g}(k)$.

On $[t_n, t_{n+1}]$ take $f(u(s)) \approx f(u(t_n))$ (independent of s), yielding

$$u^{n+1} = e^{\tau\Delta}u^n + \frac{1 - e^{\tau\Delta}}{-\Delta} f(u^n),$$

or equivalently

$$\frac{u^{n+1} - u^n}{\tau} = \frac{e^{\tau\Delta} - 1}{\tau}\,u^n + \frac{e^{\tau\Delta} - 1}{\tau\Delta}\,f(u^n). \tag{49}$$

Expansion: $(e^{\tau\Delta} - 1)/\tau = \Delta + \frac{\tau}{2}\Delta^2 + \cdots$. LHS/RHS Taylor expansions match exactly on linear terms: time differentiation produces no truncation error; but in general this alone is still not enough for an L^2 error estimate.

Theorem 22.8 (Stability; unconditional needs stabilizer)**.** *For* (49)*, multiply by* $\frac{\tau\Delta}{e^{\tau\Delta} - 1}(u^{n+1} - u^n)$ *and integrate. Set* $A = \frac{\Delta}{e^{\tau\Delta} - 1}$*; then the LHS is* $\|\sqrt{A}(u^{n+1} - u^n)\|_2^2$*, and*

$$\widehat{\sqrt{A}v}(k) = \sqrt{\frac{|k|^2}{1 - e^{-\tau|k|^2}}}\,\hat{v}(k).$$

Since $\frac{\tau|k|^2}{1 - e^{-\tau|k|^2}} \geq 1$*, one has* $\|\sqrt{A}v\|_2^2 \geq \frac{1}{\tau}\|v\|_2^2$*.*

Two RHS terms:

$$\begin{aligned} RHS_1 &= (u^n, \Delta(u^{n+1} - u^n)) = \tfrac{1}{2}\big(\|\nabla u^n\|_2^2 - \|\nabla u^{n+1}\|_2^2 + \|\nabla(u^{n+1} - u^n)\|_2^2\big), \\ RHS_2 &= (f(u^n), u^{n+1} - u^n), \qquad f(u) = u - u^3 \quad \text{(AC)}. \end{aligned}$$

Using $F(u^{n+1}) - F(u^n) = f(u^n)\delta u + \int(1 - 3s^2)(u^{n+1} - s)\,ds$ and combining yields the energy difference and $\|\sqrt{A}\delta u\|_2^2$. Stability requires $\tau \ll 1$ so that

$$\frac{1}{\tau}\|\delta u\|_2^2 \geq \tfrac{1}{2}\|\nabla\delta u\|_2^2 + \tfrac{3}{2}\max\big\{\|u^{n+1}\|_\infty^2, \|u^n\|_\infty^2\big\}\|\delta u\|_2^2$$

(e.g. $1/\tau \gg N^2 + \cdots$), or adding stabilization terms $S_1(u^{n+1} - u^n) + S_2\Delta(u^{n+1} - u^n)$.

22.9 Stabilized ETD: L^2 error

Consider (one may add $S_2(-\Delta)(u^{n+1}-u^n)$)

$$\frac{u^{n+1}-u^n}{\tau}+S_1(u^{n+1}-u^n)=\frac{e^{\tau\Delta}-1}{\tau}u^n+\frac{e^{\tau\Delta}-1}{\tau\Delta}\Pi_N f(u^n),$$
$$\frac{v^{n+1}-v^n}{\tau}+S_1(v^{n+1}-v^n)=\frac{e^{\tau\Delta}-1}{\tau}v^n+\frac{e^{\tau\Delta}-1}{\tau\Delta}\Pi_N f(v^n)+G^n.$$

$|I-\Pi_N|$ only produces an N^{-s} error tied to regularity s; we do not expand this further here.

Define the Fourier multiplier

$$\widehat{Lu}(k)=\sqrt{\frac{\tau|k|^2}{1-e^{-\tau|k|^2}}}\,\hat{u}(k),$$

satisfying $\|u\|_2\le\|Lu\|_2\le\|u\|_2+\sqrt{\tau}\|Du\|_2$.

Proposition 22.9 (L^2-error)**.** *If $\|Du^n\|_2+\|Dv^n\|_2+\|v^n\|_\infty\le M$, then*

$$\|u^m-v^m\|_L\le\exp(C_MT)\left(\|u_0-v_0\|_L+4\tau\sum_{n=0}^{m-1}\|LG^n\|_2^2\right)$$

(the norm $\|\cdot\|_L$ is compatible with L; $T=m\tau$).

Sketch. $\xi^n=u^n-v^n$. Take the inner product with $\frac{\tau Q}{e^{\tau Q}-1}\xi^{n+1}$ ($Q=\Delta$):

$$LHS=\frac{1}{2\tau}\big(\|L\xi^{n+1}\|_2^2-\|L\xi^n\|_2^2+\|L(\xi^{n+1}-\xi^n)\|_2^2\big),$$

stabilization contributions

$$LHS_2=\tfrac{S_1}{2}\big(\|L\xi^{n+1}\|_2^2-\|L\xi^n\|_2^2+\|L\delta\xi\|_2^2\big),$$
$$LHS_3=\tfrac{S_2}{2}\big(\|L\nabla\xi^{n+1}\|_2^2-\|L\nabla\xi^n\|_2^2+\|L\nabla\delta\xi\|_2^2\big)$$

(one may omit S_i first in the analysis). Split the RHS as (i) + (ii) + (iii):

$$\text{(i)}=(\xi^n,\Delta\xi^{n+1})=-\tfrac12\big(\|\nabla\xi^{n+1}\|_2^2+\|\nabla\xi^n\|_2^2\big)+\tfrac12\|\nabla\delta\xi\|_2^2,$$
$$\text{(ii)}=(f(u^n)-f(v^n),\xi^{n+1}),\quad f(u)-f(v)=(a_1+a_2(v^n)^2)\xi+a_3v(\xi)^2+a_4(\xi)^3$$

(a_i can be computed exactly). Young estimates absorb into $\frac18\|\xi^{n+1}\|_2^2$, etc.; (iii) contains (LG_n,ξ^{n+1}). After combining,

$$\tfrac12\big(\|L\xi^{n+1}\|_2^2-\|L\xi^n\|_2^2\big)+\big(S_1-\tfrac12\big)\Delta_n\|L\xi\|_2^2$$
$$+\big(S_2-\tfrac12\big)\Delta_n\|L\nabla\xi\|_2^2\le C\|L^2G_n\|_2^2+C\|L\xi^n\|_2^2.$$

Discrete Grönwall applies; $S_2>\frac12$ suffices. □

Truncation: $Q_n=-\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_t\Delta u\cdot(t_{n+1}-t)\,dt+\cdots$, $\|Q_n\|_2\lesssim\tau\|\partial_t\Delta u\|_{L_t^\infty L_x^2}$, hence $\sum\tau\|Q_n\|_2^2\lesssim\tau\cdot t_m$, and thus $\|u^{n+1}-u(t_{n+1})\|_2\lesssim\tau$ (first-order scheme).

22.10 Stabilized exponential integrator for CH

(See slides on CH.) Exponential scheme

$$u^{n+1}=e^{-\nu\Delta^2\tau}u^n+\frac{1-e^{-\nu\Delta^2\tau}}{\nu\Delta}\Pi_N f(u^n)-S\tau\Delta^2(u^{n+1}-u^n),\tag{50}$$

where the last term is the stabilizer. Pros: regularity, accuracy, easy to compute; cons: stability depends on S.

22.11 Incompressible NS / Euler on $\mathbb{T}^2$

Incompressible Navier–Stokes / Euler:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u, \\ u(0,x) = u_0(x), \qquad \operatorname{div} u = 0, \end{cases} \qquad \text{(NS; } \nu\text{: viscosity)}, \tag{51}$$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = 0, \\ u(0,x) = u_0(x), \qquad \operatorname{div} u = 0, \end{cases} \qquad \text{(Euler)}. \tag{52}$$

Reynolds number $Re = LU/\nu$ (dimensionless): $Re > 4000$ turbulent, $Re < 2000$ laminar. Questions: vanishing viscosity? stability?

Divergence form and vorticity

If $\operatorname{div} u = 0$, then $\operatorname{div}(u\otimes u) = (u\cdot\nabla)u$. Vorticity: in 3D, $\omega = \nabla\times u$; in 2D, the scalar $\omega = \nabla^\perp\cdot u$, $\nabla^\perp = (\partial_2, -\partial_1)$. Applying $\nabla^\perp\cdot$ to 2D NS, since $\nabla^\perp\cdot\nabla p = 0$ and $\nabla^\perp\cdot((u\cdot\nabla)u) = (u\cdot\nabla)\omega$, one gets

$$\partial_t\omega + (u\cdot\nabla)\omega = \nu\Delta\omega.$$

Example 22.10 (Taylor–Green vortex)**.** On $\mathbb{T}^2$ the field

$$\omega(t,x,y) = 2\cos x\cos y\, e^{-2\nu t}$$

is an exact Navier–Stokes solution (the nonlinear term vanishes). A spectral backward Euler step on the vorticity equation tracks the decay; the spatial error is spectral, so the observed rate is first-order in Δt.

Viscosity approximation / semi-implicit scheme

Viscous approximation ($\mathbb{T}^d$ periodic; see slides): large viscosity is smooth and stable; small viscosity produces vortices; focus on the vanishing-viscosity rate. Semi-implicit scheme

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} + u^n\cdot\nabla u^{n+1} + \nabla p^{n+1} = \nu\Delta u^{n+1}, \\ \operatorname{div} u^{n+1} = 0. \end{cases}$$

An L^2 error needs $\partial_{tt}u \in L^2$ (truncation), corresponding to $\Delta^2 u \in L^2$, i.e. $u \in H^4$; the nonlinearity further requires $\partial_t[(u\cdot\nabla)u] \in L^2$, heuristically needing $\|\Delta u\|_2\|\nabla u\|_\infty$, etc., still pointing to $u \in H^4$.

22.12 Leray projection and exponential NS schemes

Definition 22.11 (Leray projection)**.** $\mathbb{P}f = f - \nabla q$, where $\Delta q = \operatorname{div} f$ ($q = (\operatorname{div} f)/\Delta$). Then $\operatorname{div}(\mathbb{P}f) = 0$ (removing the $\operatorname{div} \neq 0$ part). Properties: $\mathbb{P}u = u$, $\mathbb{P}\nabla p = 0$, $\mathbb{P}\partial_t u = \partial_t u$.

Projected NS: $\partial_t u + \mathbb{P}(u\cdot\nabla u) = \nu\Delta u$. Duhamel (normal exponential method):

$$u(t_{n+1}) = e^{\tau\nu\Delta}u(t_n) - \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}(u\cdot\nabla u)(s)\,ds.$$

Naive freezing $u(s) \approx u(t_n)$

From NS, $u(s) = u(t_n) + \nu\int_{t_n}^s \Delta u - \int_{t_n}^s \mathbb{P}(u\cdot\nabla u)$, one gets

$$u(t_{n+1}) = e^{\tau\nu\Delta}u(t_n) - \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}\big(u(t_n)\cdot\nabla u(t_n)\big)\,ds + R_n,$$

$$R_n = -\int e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}\big[u(s)\cdot\nabla u(s) - u(t_n)\cdot\nabla u(t_n)\big]\,ds.$$

Substituting $u(s) - u(t_n)$ requires at least $u \in H^3$.

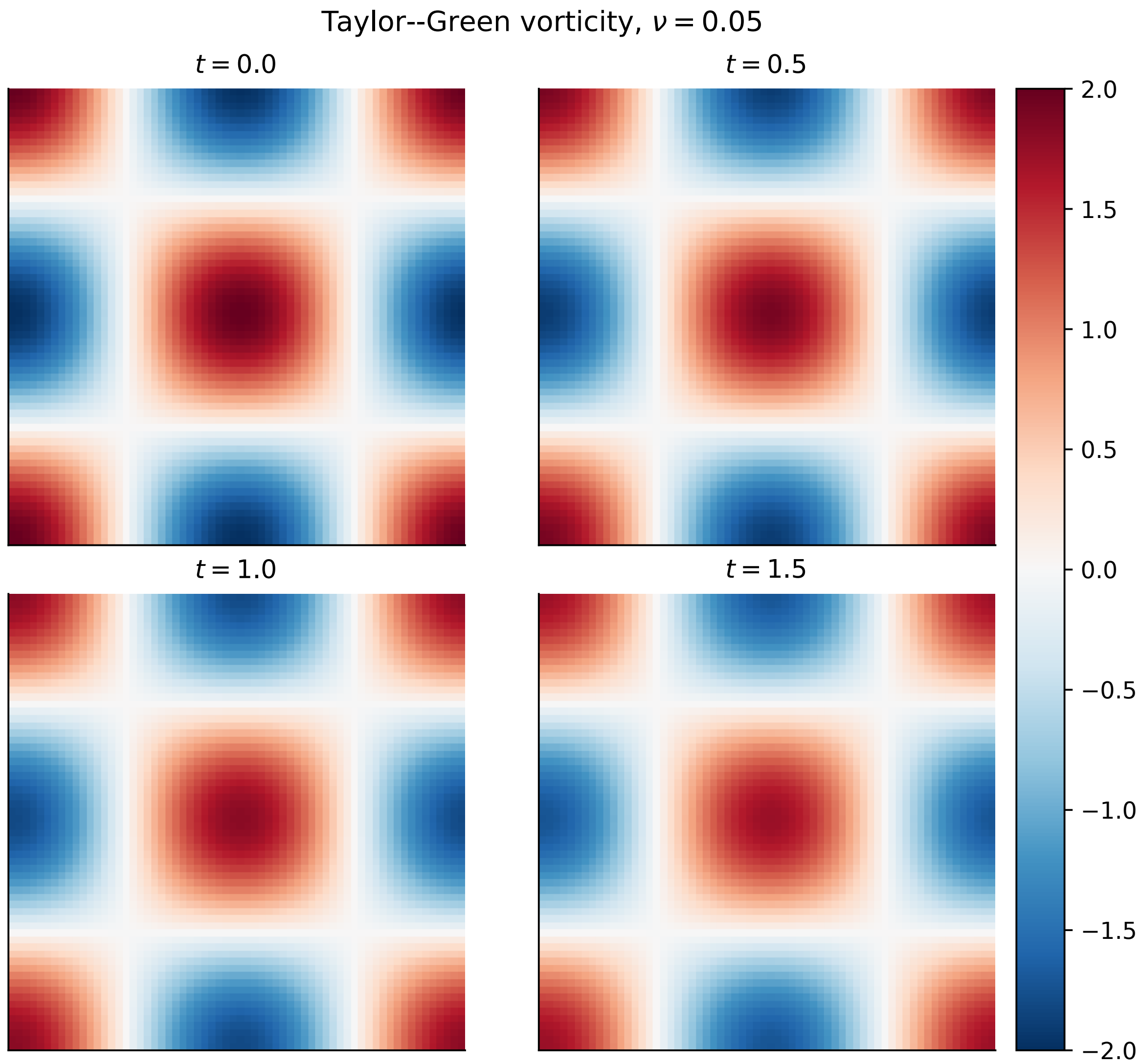


Figure 13: Taylor–Green vorticity at $\nu = 0.05$.

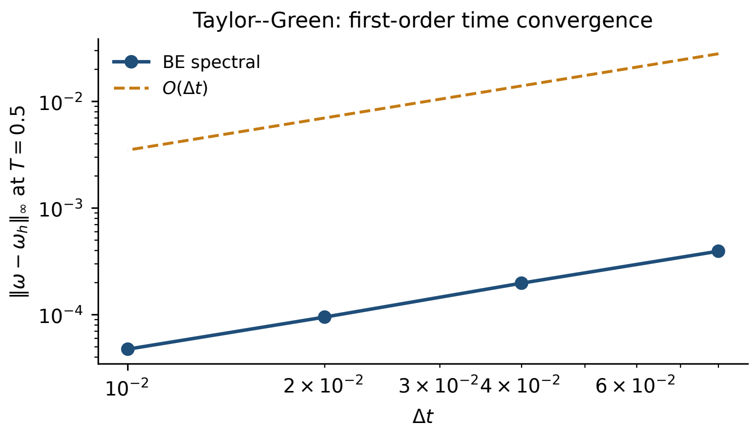


Figure 14: Max-norm error at $T = 0.5$ versus Δt for spectral backward Euler.

New idea: freeze along heat flow

Approximate $u(s)$ by $e^{(s-t_n)\nu\Delta}u(t_n)$:

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}(u(s)\cdot\nabla u(s))\,ds\\ &= \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}\big(e^{(s-t_n)\nu\Delta}u(t_n)\cdot\nabla e^{(s-t_n)\nu\Delta}u(t_n)\big)\,ds - R_{n,1}.\end{aligned}$$

The two terms I_1, I_2 in $R_{n,1}$ give $\|R_{n,1}\|_2 \lesssim \tau^2\|u\|_{H^2}^3$ (at most needing $u\in H^2$).

Write $V(s) = e^{(s-t_n)\nu\Delta}u(t_n)$, $g(s) = e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}(V(s)\cdot\nabla V(s))$. Use $g(s) = g(t_{n+1}) - \int_s^{t_{n+1}} g'(\sigma)\,d\sigma$. Since $\Delta\mathbb{P} = \mathbb{P}\Delta$, the Fourier definition (2D)

$$\widehat{\mathbb{P}\begin{pmatrix} f\\ g\end{pmatrix}}(k) = \begin{pmatrix}\hat f - \frac{k_1\hat f + k_2\hat g}{|k|^2}k_1\\ \hat g - \frac{k_1\hat f+k_2\hat g}{|k|^2}k_2\end{pmatrix},$$

verifies $\operatorname{div}\mathbb{P} = 0$. Computation yields

$$g'(s) = \nu e^{(t_{n+1}-s)\nu\Delta}\mathbb{P}\Big(-\sum_j \partial_j V\cdot\partial_j\nabla V\Big),$$

hence $\|g'(s)\|_2 \lesssim \nu\|V\|_{W^{2,2+\varepsilon}}^2$, so $\|R_{n,2}\|_2 \lesssim \nu\tau^2\|u\|_{W^{2,2+\varepsilon}}^2$, and

$$\int_{t_n}^{t_{n+1}} g(t_{n+1})\,ds = \tau\mathbb{P}\big(e^{\nu\tau\Delta}u(t_n)\cdot\nabla e^{\nu\tau\Delta}u(t_n)\big).$$

Implicit gradient for stability

The scheme $u^{n+1} = e^{\tau\Delta}u^n - \tau\mathbb{P}(e^{\tau\Delta}u^n\cdot\nabla e^{\tau\Delta}u^n)$ loses stability as $\tau\to 0$ (accuracy disappears). Switch to ∇u^{n+1}:

$$u^{n+1} = e^{\tau\Delta}u^n - \tau\mathbb{P}\big(e^{\tau\Delta}u^n\cdot\nabla u^{n+1}\big). \tag{53}$$

The exact solution satisfies $u(t_{n+1}) = e^{\tau\Delta}u(t_n) - \tau\mathbb{P}(e^{\tau\Delta}u(t_n)\cdot\nabla u(t_{n+1})) - R_{n,1} - R_{n,2} - \mathbb{P}R_{n,3}$, where

$$R_{n,3} = \tau e^{\tau\Delta}u(t_n)\cdot\nabla\big(e^{\tau\Delta}u(t_n) - u(t_{n+1})\big)$$

(written via Duhamel as an integral), $\|R_{n,3}\|_2 \lesssim \tau^2\|u\|_{H^2}^3$. Summary: local truncation $\|R_{n,1}\|_2 + \|R_{n,2}\|_2 + \|R_{n,3}\|_2 \lesssim \tau^2$, with regularity needs H^2, $W^{2,2+\varepsilon}$, H^2 respectively.

Theorem 22.12 (Energy stability)**.** *For* (53)*, if* $\operatorname{div} u^{n+1} = 0$*, then* $\|u^{n+1}\|_2 \le \|u^n\|_2$.

Proof. Take the inner product with u^{n+1}. Since $(\mathbb{P}f, u^{n+1}) = (f, u^{n+1})$ and $\operatorname{div}(e^{\tau\Delta}u^n) = 0$,

$$\int_{\mathbb{T}^2} e^{\tau\Delta}u^n\cdot\nabla\tfrac12|u^{n+1}|^2\,dx = 0.$$

Hence $\|u^{n+1}\|_2^2 = (e^{\tau\Delta}u^n, u^{n+1}) \le \|u^n\|_2\|u^{n+1}\|_2$. □

L^2 error

Set $e^n = u^n - u(t_n)$. Error equation $(*)$:

$$\begin{aligned}e^{n+1} + \tau\mathbb{P}(e^{\tau\Delta}u^n\cdot\nabla e^{n+1}) &= e^{\tau\Delta}e^n - \tau\mathbb{P}(e^{\tau\Delta}e^n\cdot\nabla u(t_{n+1}))\\ &\quad + R_{n,1} + R_{n,2} + \mathbb{P}R_{n,3}.\end{aligned}$$

Test with e^{n+1}: $LHS = \|e^{n+1}\|_2^2$,

$$\begin{aligned}RHS &\le \tfrac12\|e^n\|_2^2 + \tfrac12\|e^{n+1}\|_2^2 + C\tau\|\nabla u(t_{n+1})\|_\infty\|e^n\|_2\|e^{n+1}\|_2 + C\tau^2\|e^{n+1}\|_2\\ &\le \tfrac12\|e^n\|_2^2 + \tfrac12\|e^{n+1}\|_2^2 + C\tau\|e^n\|_2^2 + C\tau\|e^{n+1}\|_2^2 + C\tau^3\end{aligned}$$

(needs $u\in W^{1,\infty}$). Rearranging $(1-C\tau)\|e^{n+1}\|_2^2 \le (1+C\tau)\|e^n\|_2^2 + C\tau^3$, discrete Grönwall $\Rightarrow \|e^n\|_2^2 \lesssim \tau^2$. Overall one still needs $u\in W^{2,2+\varepsilon}$ (from $R_{n,2}$).

23 Further remarks on phase-field schemes and fluids

(Source: course slides together with the München and Feishu talks. Phase-field definitions, sharp-interface limits, Fourier spectral basics, and exponential-integrator Duhamel derivations appear in earlier sections of this part; here we keep only non-overlapping scheme formulas, comparison tables, and fluid/SQG highlights.)

23.1 Energy stability as a design criterion

Gradient-flow structure of Allen–Cahn / Cahn–Hilliard implies continuous energy dissipation, but discretization need not inherit it. The slides treat *energy-dissipative* time schemes as the main figure of merit, and organize methods as: convex splitting; stabilized semi-implicit (IMEX / BDF2); operator splitting; (low-regularity) exponential integrators; fully implicit / backward Euler for interface tracking.

23.2 Stabilized semi-implicit schemes

With time step $\tau > 0$, stabilization $A > 0$, and Fourier truncation Π_N:

$$\frac{u^{n+1} - u^n}{\tau} = \nu \Delta u^{n+1} - A(u^{n+1} - u^n) - \Pi_N f(u^n), \tag{54}$$

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu \Delta u^{n+1} - A\tau(u^{n+1} - u^n) - \Pi_N \big(2f(u^n) - f(u^{n-1})\big). \tag{55}$$

Unconditional energy dissipation holds if $A \geq C \cdot h(\nu)$, with $h(\nu)$ as in Table 1 (BDF2 dissipates a *modified* energy).

Table 1: Stabilization $A \geq C \cdot h(\nu)$ for unconditional energy dissipation.

$h(\nu)$	1st-order IMEX	2nd-order BDF2
2D	$\nu^{-1}\lvert\log\nu\rvert$	$\nu^{-10}\lvert\log\nu\rvert^4$
3D	ν^{-3}	ν^{-8}

Remark 23.1 (Typical numerics on the slides)**.** 2D AC with BDF2: $\nu = 0.01$, $A = 1$, $\tau = 0.01$, $N = 256$, seven-ball u_0 (coarsening of small droplets). 2D CH: $\nu = 0.009$, $A = 10$, same grid/time step.

23.3 Fully implicit radial Allen–Cahn and interface iteration

Radial fully implicit backward Euler:

$$\frac{u_{n+1} - u_n}{\tau} = \partial_{rr} u_{n+1} + \frac{1}{r}\partial_r u_{n+1} - \frac{1}{\varepsilon^2} f(u_{n+1}).$$

Theorem 23.2 (C–Li–Promislow–Wetton 2021)**.** *Fix $1 < s \leq 2$ and the distinguished limit $\tau = \varepsilon^s$. For suitable small perturbations of a radial interface of radius $R_0 > 1$, there is an interface sequence $\{R_n\}$ obeying*

$$\frac{R_{n+1} - R_n}{\tau} = -\frac{1}{R_{n+1}} + \frac{b_1 (R_{n+1} - R_n)^3}{\tau \varepsilon^2} + O\Big(\frac{\tau}{\varepsilon}\Big), \qquad b_1 = \frac{\|g''\|_{L^2}^2}{6\|g'\|_{L^2}^2} > 0,$$

until the interface crosses the threshold $R = 1$.

23.4 Strang operator splitting

For AC, split linear diffusion and the ODE $u_t = u - u^3$:

$$\mathcal{S}_{\mathcal{L}}(\tau)a = \exp(\varepsilon^2\tau\Delta)a, \qquad \mathcal{S}_{\mathcal{N}}(\tau)b = \big((e^{2\tau}-1)b^2+1\big)^{-1/2}e^{\tau}b.$$

Strang composition:

$$u^{n+1} = \mathcal{S}_{\mathcal{L}}(\tau/2)\,\mathcal{S}_{\mathcal{N}}(\tau)\,\mathcal{S}_{\mathcal{L}}(\tau/2)\,u^n,$$

equivalently

$$u^{n+1} = e^{\varepsilon^2\frac{\tau}{2}\Delta}\left(\frac{e^{\tau}\,e^{\varepsilon^2\frac{\tau}{2}\Delta}u^n}{\big[1+(e^{2\tau}-1)\big(e^{\varepsilon^2\frac{\tau}{2}\Delta}u^n\big)^2\big]^{1/2}}\right).$$

Energy decrease $E(u^{n+1}) \le E(u^n)$ is nontrivial (explicit energy map is complicated). For CH, the naive nonlinear step $\partial_t u = -\Delta(u-u^3)$ has no usable explicit solver and is not even well-posed; a modified splitting yields uniform Sobolev bounds $\sup_n \|u^n\|_{H^k} \le C$ (rather than a direct energy identity).

23.5 SQG, Onsager, and convex integration

Dissipative SQG on $\mathbb{T}^2$:

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = -\nu\Lambda^{\gamma}\theta, \\ u = \nabla^{\perp}\Lambda^{-1}\theta, \qquad \Lambda = (-\Delta)^{1/2}, \quad \nu\ge 0,\ 0<\gamma\le 2, \\ \theta|_{t=0} = \theta_0, \qquad \displaystyle\int_{\mathbb{T}^2}\theta = 0. \end{cases} \tag{56}$$

Onsager-type question: lowest regularity for which conservation persists. Convex integration (Gromov; De Lellis–Székelyhidi; …) constructs rough solutions; for time-dependent SQG the *odd-multiplier obstruction* for plane-wave ansatz is removed by a momentum / potential-velocity reformulation $u = \Lambda v$ (Buckmaster–Shkoller–Vicol; Resnick), yielding solutions with $\Lambda^{-1}\theta \in C^{\beta}$, $\beta \in (\frac12, \frac45)$ (Onsager range $\beta \in (\frac12, 1)$).

Theorem 23.3 (C–Kwon–Li 2021, stationary SQG)**.** *For any $\nu \ge 0$, $\gamma \in (0, \frac32)$, and $\frac12 < \alpha < \frac12 + \min(\frac16, \frac32 - \gamma)$, there exist infinitely many zero-mean stationary weak solutions with $\Lambda^{-1}\theta \in C^{\alpha}(\mathbb{T}^2)$.*

Numerical tests of a low-regularity exponential integrator for (56) (convergence in τ, vanishing-viscosity rate, vortex waltz, and a rough-data cascade) are given in Section 24.5.

23.6 Stabilized exponential integrator for CH

(The standard Allen–Cahn ETD from Duhamel appears earlier in this part.) For CH the slides emphasize a stabilized scheme (analytic results previously scarce: no maximum principle):

$$u^{n+1} = e^{-\nu\Delta^2\tau}u^n + \frac{1-e^{-\nu\Delta^2\tau}}{\nu\Delta}\Pi_N[f(u^n)] - S\tau\Delta^2(u^{n+1}-u^n), \qquad S>0.$$

With seven-bubble data, $\nu = 0.01$, $\tau = 0.001$ and $N = 128$, the choice $S = 0$ blows up by time $t = 1$; a small $S > 0$ still oscillates at $t = 20$; a sufficiently large S yields a clean evolution (Figure 18). A manufactured solution $u_e = \frac12 e^{-t}\sin x \sin y$ (with the corresponding forcing) confirms first-order temporal rates in L^2 and L^∞ (Table 2).

23.7 Incompressible NS / Euler schemes

On $\mathbb{T}^2$, Euler and Navier–Stokes are the usual inviscid / viscous systems. The inviscid limit $\nu \to 0$ holds in Sobolev spaces under suitable assumptions (e.g. Masmoudi: H^{s-2} convergence for H^s data, $s > 2$), but *need not* be mirrored by a given discretization.

Semi-implicit spectral scheme (C.–Luo–Wang 2024):

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} + \Pi_N(u^n \cdot \nabla u^{n+1}) + \nabla p^n = \nu \Delta u^{n+1}, \quad \operatorname{div} u^{n+1} = 0, \\ u^0 = \Pi_N u_0. \end{cases} \tag{57}$$

Unconditional L^2 dissipation $\|u^{n+1}\|_{L^2} \le \|u^n\|_{L^2}$ and $\sup_n \|u^n\|_{H^s} \le 4\|u_0\|_{H^s}$; errors of the form

$$\sup_n \|u^{n+1} - v(t_{n+1})\|_{L^2} \le C_1(\tau + N^{-s+1}),$$

and, for $u_0 \in H^{s+3}$,

$$\sup_n \|u^{n+1} - v(t_{n+1})\|_{H^s} \le C_2\big(\tau + N^{-2} + N^2\nu\big).$$

An iterative Fourier realization with Leray projector $\mathbb{P}$ converges when $\tau \le C^* \max\{\nu, N^{-1}\}$.

Remark 23.4 (Experiments highlighted on the slides)**.** Two Gaussian vortices at $T = 50$ ($\tau = 0.001$, $N = 128$): small ν yields filamentation / merger; large ν yields a smooth blob. High-order errors are subtle: the $N^2\nu$ term means refining N at fixed ν can *increase* H^8 error even while lower norms improve. Two-vortex dynamics and τ/ν refinement plots appear in Section 24.4.

24 Further numerical examples

The computations below reuse the original figures from `slides/file` (WCCM–ECCOMAS 2026, München) and `slides/files` (Feishu Institute, 2026).

24.1 Dendrites, seven-bubble AC / CH, and a CH exponential integrator

A classical motivation for phase-field models is dendritic solidification: a diffuse interface of width ε approximates a free boundary whose law is not elementary (Figure 15).

On $\mathbb{T}^2$, the IMEX / BDF2 schemes of Section 23.2 with seven-ball initial data produce curvature coarsening for Allen–Cahn and mass-conserving Ostwald ripening for Cahn–Hilliard (Figures 16–17).

The stabilized exponential integrator (50) is more delicate. With seven-bubble data, $\nu = 0.01$, $\tau = 0.001$ and $N = 128$, the unstabilized scheme $S = 0$ blows up by time $t = 1$; a small S still oscillates at $t = 20$; a sufficiently large S yields a clean evolution (Figure 18). A manufactured solution $u_e = \frac{1}{2}e^{-t}\sin x \sin y$ (with the corresponding forcing) recovers a first-order rate in τ (Table 2; the column "ratio" is the successive error ratio, consistently ≈ 2).

Table 2: Manufactured Cahn–Hilliard test $u_e = \frac{1}{2}e^{-t}\sin x \sin y$ (so $u_0 = \frac{1}{2}\sin x \sin y$), $\nu = 1$, $S = 0.1$, $N = 256$. Successive error ratios ≈ 2 indicate first-order accuracy in τ.

τ	L^2 error	ratio	L^∞ error	ratio
0.01	9.099×10^{-5}	—	1.156×10^{-3}	—
/2	4.523×10^{-5}	2.012	5.745×10^{-4}	2.012
/4	2.255×10^{-5}	2.006	2.864×10^{-4}	2.006
/8	1.126×10^{-5}	2.003	1.430×10^{-4}	2.003
/16	5.624×10^{-6}	2.002	7.143×10^{-5}	2.002
/32	2.811×10^{-6}	2.001	3.570×10^{-5}	2.001

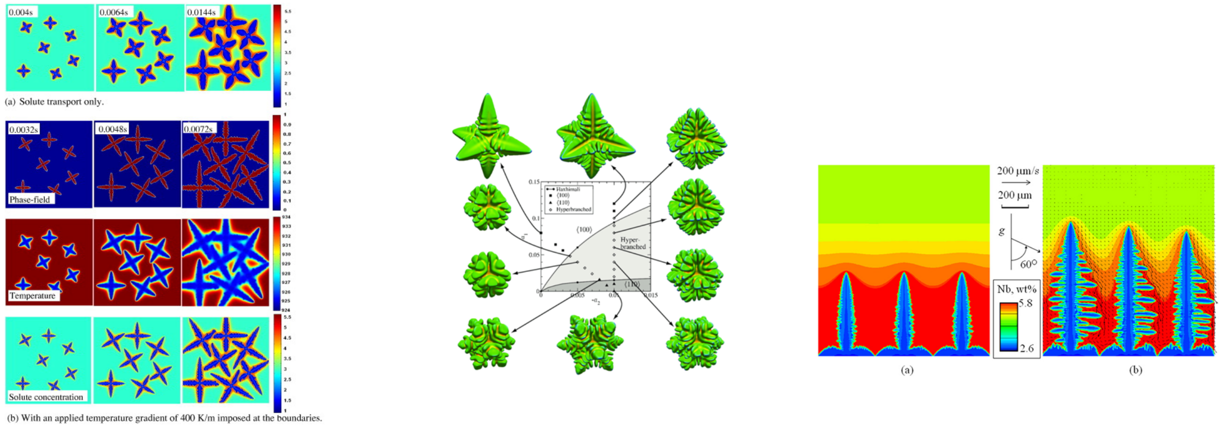


Figure 15: Dendritic growth: a phase-field interface as a surrogate for a sharp free boundary.

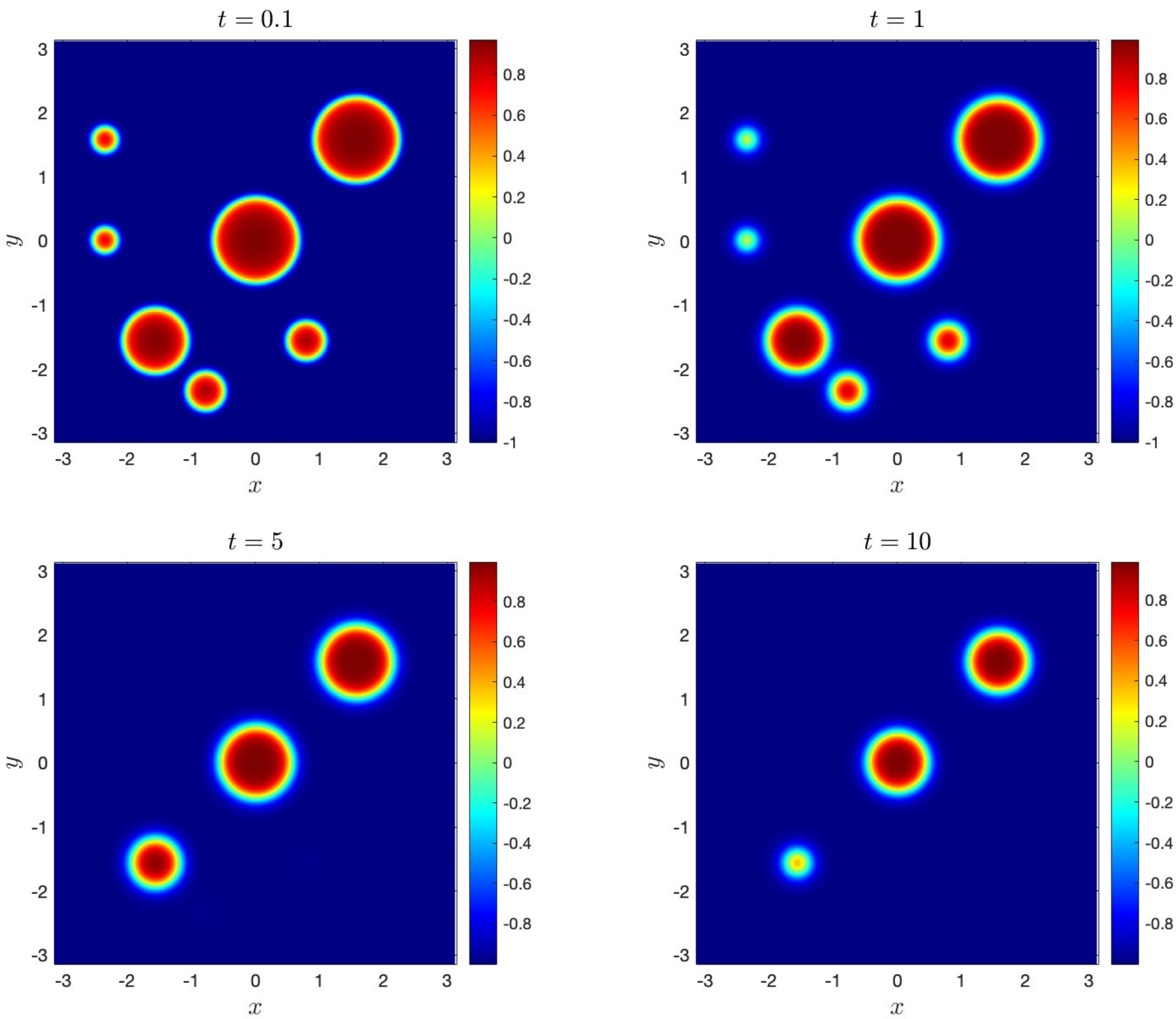


Figure 16: 2D Allen–Cahn, BDF2, seven-ball data: $\nu = 0.01$, $A = 1$, $\tau = 0.01$, $N = 256$, times $t = 0.1, 1, 5, 10$.

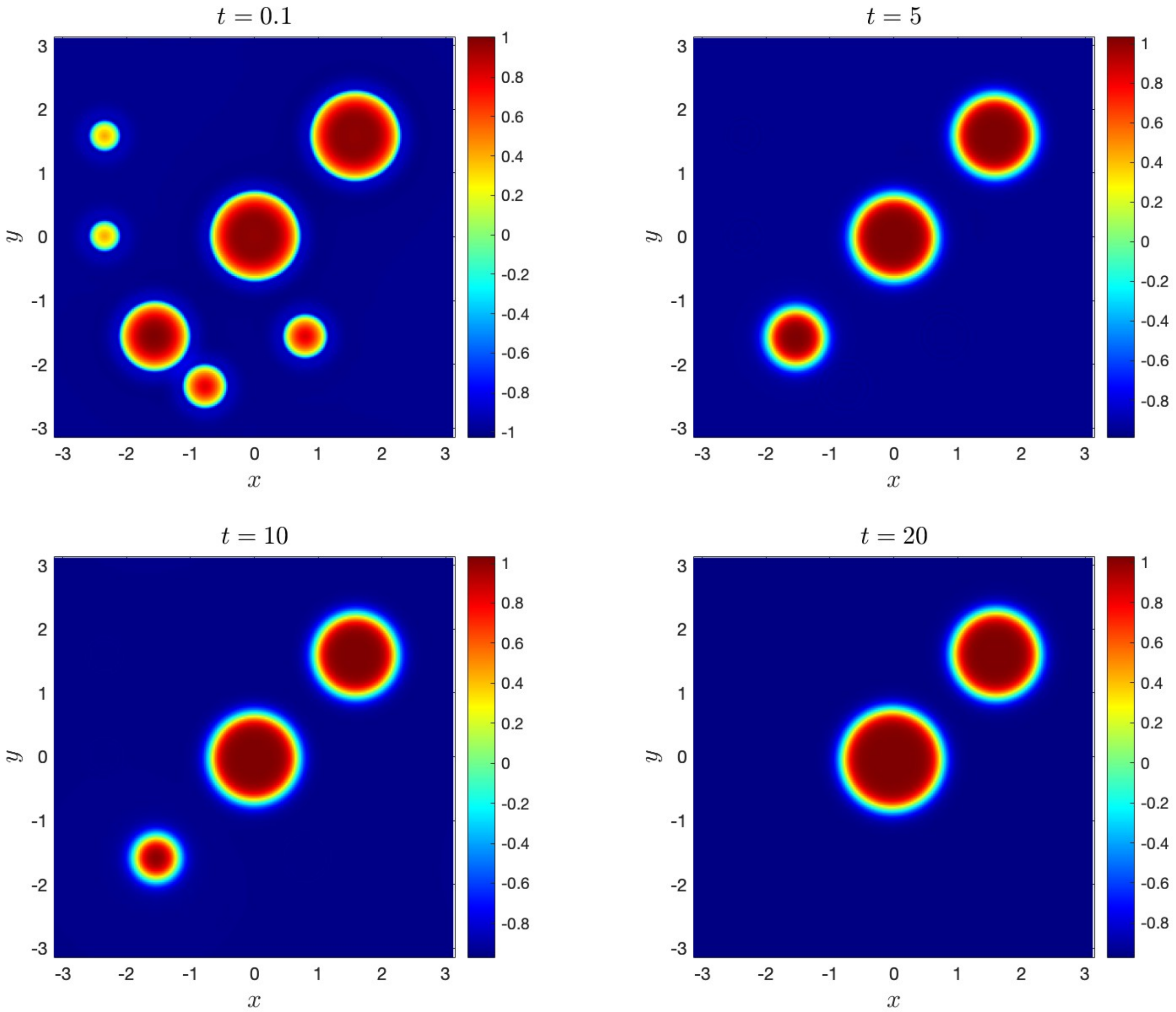
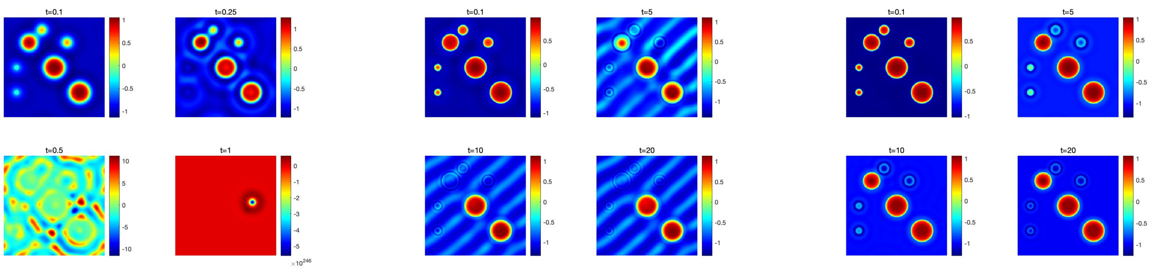


Figure 17: 2D Cahn–Hilliard, seven-ball data: $\nu = 0.009$, $A = 10$, $\tau = 0.01$, $N = 256$, times $t = 0.1, 5, 10, 20$.



$S = 0$ (blow-up) small S (oscillatory) large S (stable)

Figure 18: Stabilized LEI for Cahn–Hilliard, seven-bubble data, $\nu = 0.01$, $\tau = 0.001$, $N = 128$.

24.2 Parabolic sine–Gordon

The parabolic sine–Gordon equation

$$\partial_t u = \nu \Delta u + \sin u \tag{58}$$

is an L^2 gradient flow of $E(u) = \int \big(\frac{\nu}{2}|\nabla u|^2 + \cos u\big)\, dx$. The nonlinearity $\sin u = u - \frac{1}{6}u^3 + \cdots$ is a bounded analogue of the double well. First-order IMEX and BDF2,

$$\frac{u^{n+1} - u^n}{\tau} = \nu \Delta u^{n+1} + \sin(u^n), \qquad \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu \Delta u^{n+1} + 2\sin(u^n) - \sin(u^{n-1}),$$

inherit energy stability and a discrete maximum principle. Figures 19–21 compare BDF2 dynamics and energy decay against Allen–Cahn with the same integrator.

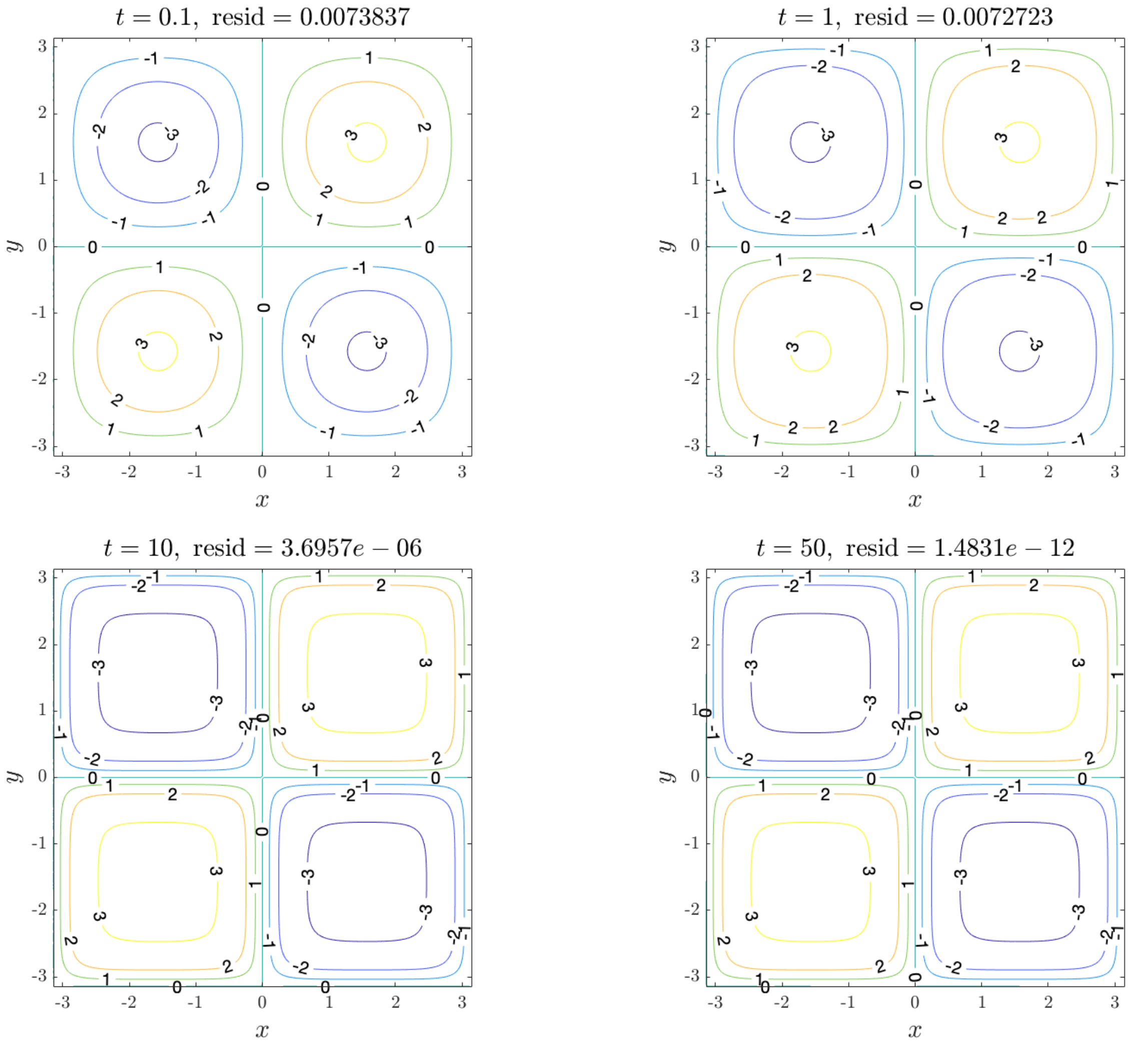


Figure 19: Parabolic sine–Gordon, BDF2: $\nu = 0.04$, $u_0 = \pi \sin x \sin y$, $\tau = 0.01$, $N = 256$.

24.3 Molecular beam epitaxy

Epitaxial growth with slope selection is the L^2 gradient flow

$$\partial_t h = -\nu \Delta^2 h + \nabla \cdot f(\nabla h), \qquad f(z) = z\big(|z|^2 - 1\big), \tag{59}$$

with energy $E_{\mathrm{MBE}}(h) = \int \big(\frac{\nu}{2}|\Delta h|^2 + \frac{1}{4}(|\nabla h|^2 - 1)^2\big)\, dx$. A sinc-type variant

$$\partial_t h = -\nu \Delta^2 h - \nabla \cdot \big(\mathrm{sinc}(|\nabla h|)\, \nabla h\big)$$

dissipates $\int \big(\frac{1}{2}\nu|\Delta h|^2 + \cos(|\nabla h|)\big)\, dx$. IMEX Euler and BDF2 with $g(z) = -\mathrm{sinc}(|z|)z$ remain energy stable.

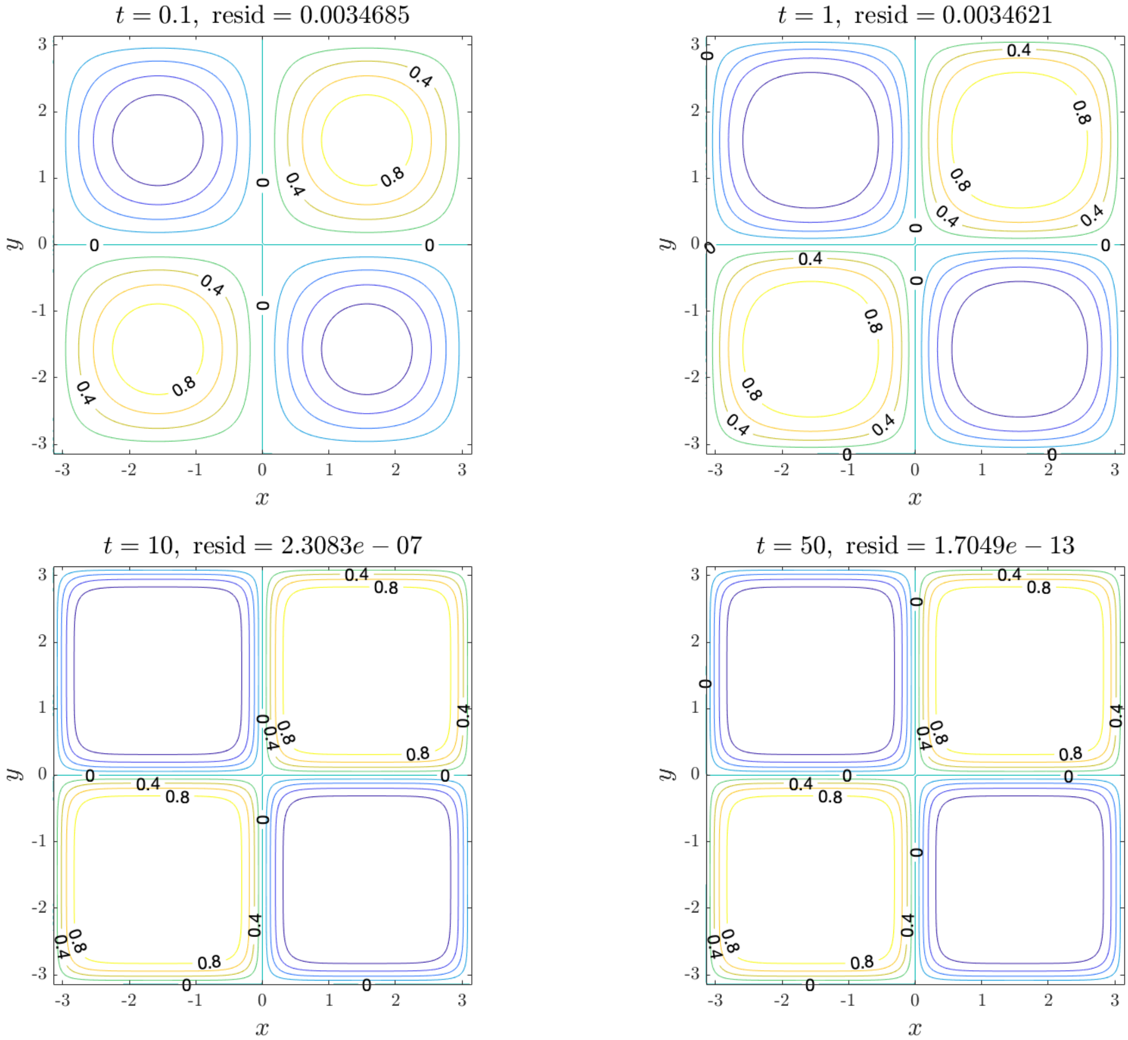


Figure 20: Allen–Cahn with the same BDF2 integrator: $\nu = 0.04$, $u_0 = \sin x \sin y$.

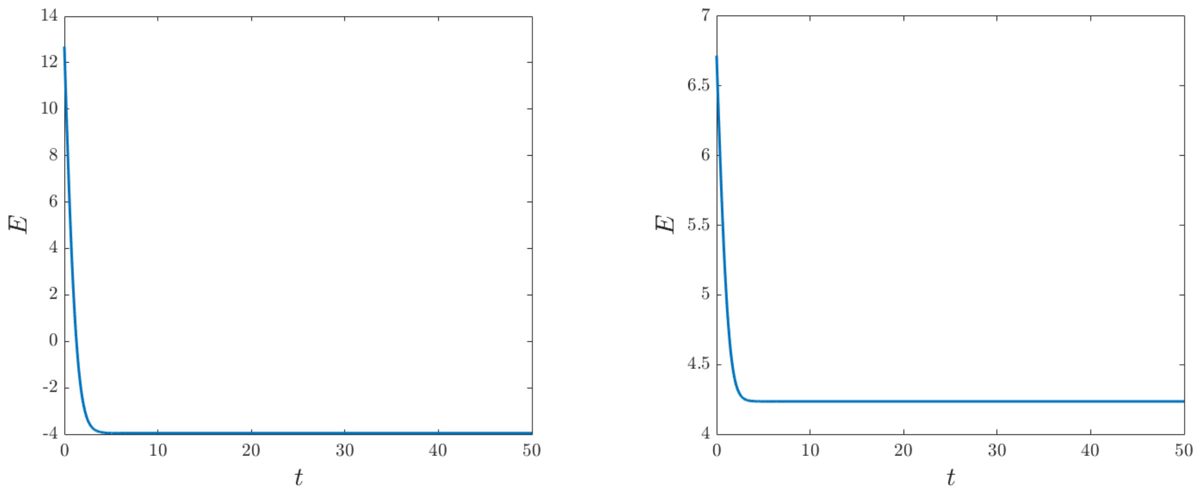


Figure 21: Energy histories: sine–Gordon (left) and Allen–Cahn (right).

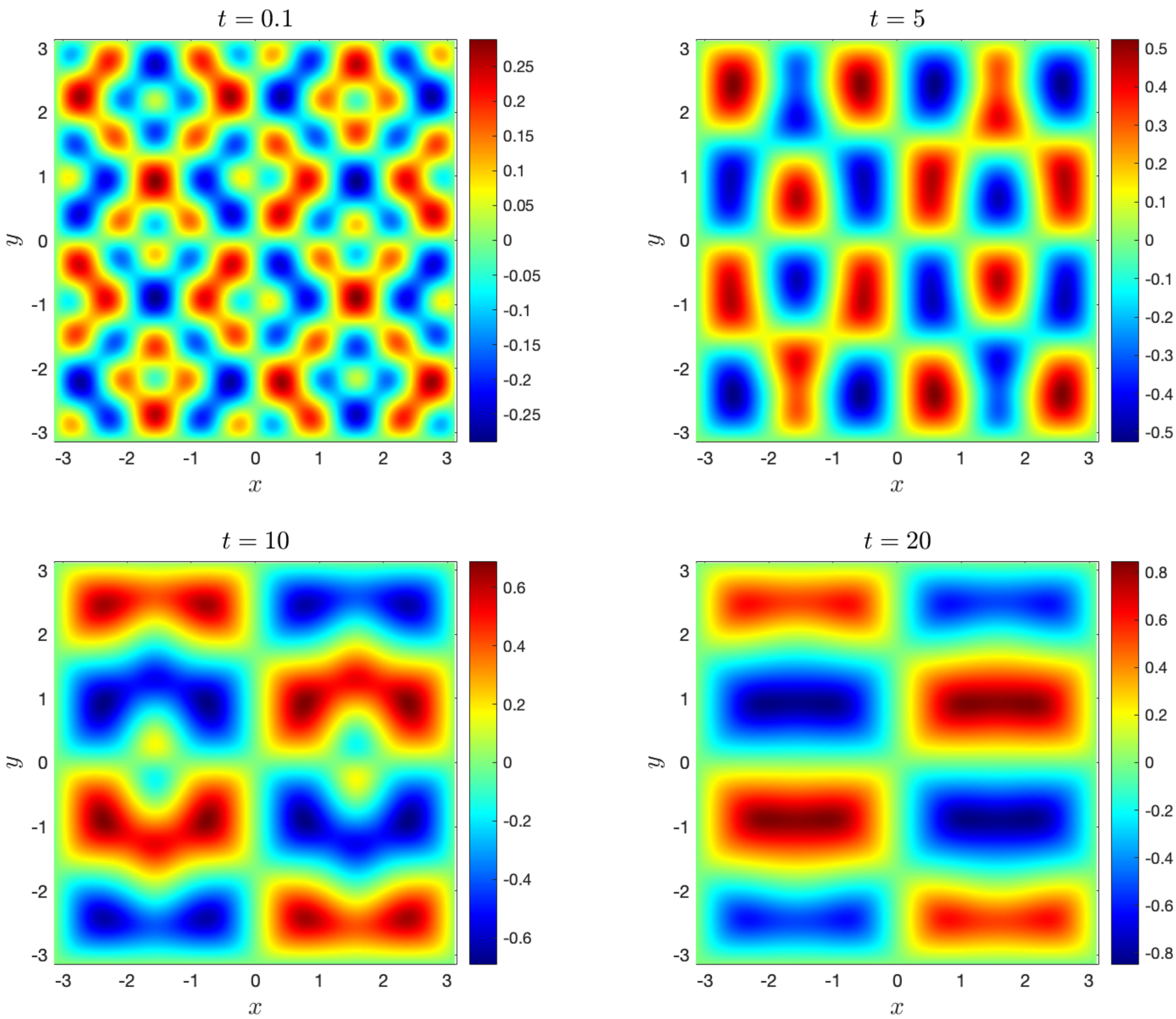


Figure 22: Classical MBE (59), first-order IMEX: $\nu = 0.01$, $\tau = 0.001$, $N = 256$.

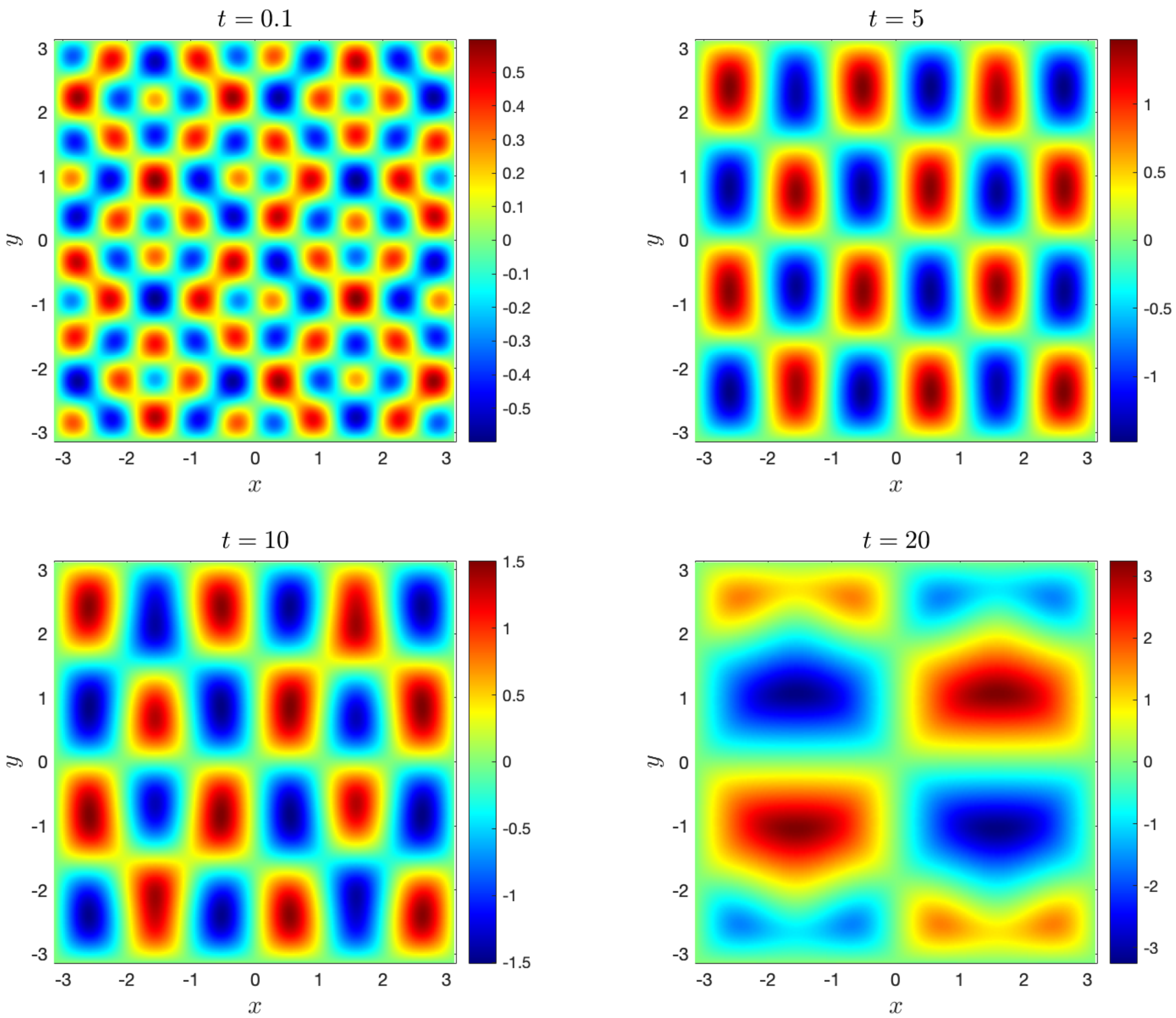


Figure 23: Sinc-type MBE, same grid and time step.

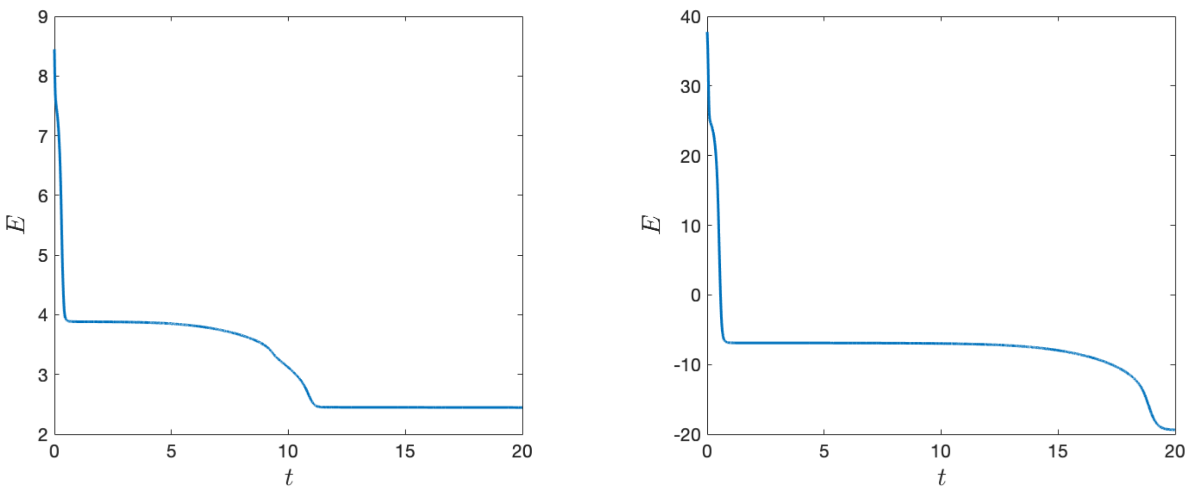


Figure 24: Energy decay for classical (left) and sinc-type (right) MBE.

24.4 Two-vortex Navier–Stokes

Beyond the Taylor–Green eigenmode, a standard nonlinear test is a pair of Gaussian vortices

$$\omega_0 = \exp\big(-5\big((x+\tfrac{\pi}{4})^2 + y^2\big)\big) + \exp\big(-5\big((x-\tfrac{\pi}{4})^2 + y^2\big)\big).$$

The semi-implicit spectral scheme of Section 23.7 at $T = 50$, $\tau = 0.001$, $N = 128$ yields merger and filamentation when ν is small, and a single smooth blob when ν is large (Figure 25). A manufactured solution

$$u_e(t,x,y) = e^{-t}\big(-\tfrac{1}{2}\sin x\cos y,\ \tfrac{1}{2}\cos x\sin y\big)$$

gives first-order errors in τ at fixed ν; refining ν at fixed τ is recorded in Figure 26.

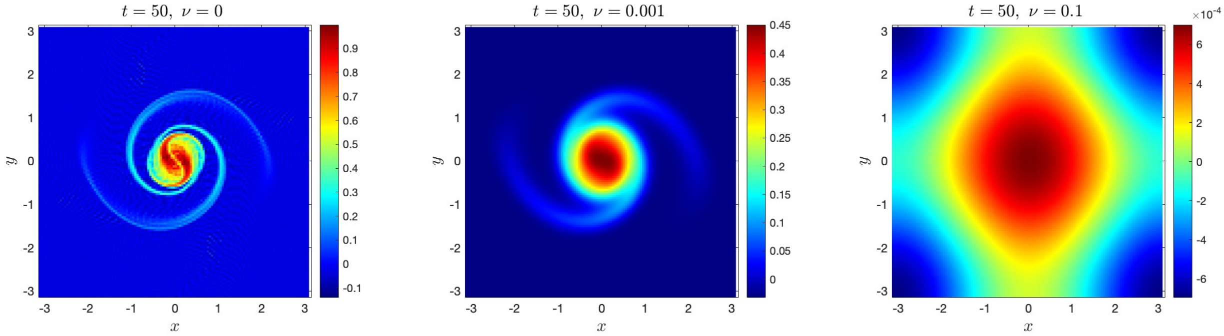


Figure 25: Two-vortex Navier–Stokes at $T = 50$ for $\nu = 0$, 0.001, and 0.1 ($\tau = 0.001$, $N = 128$).

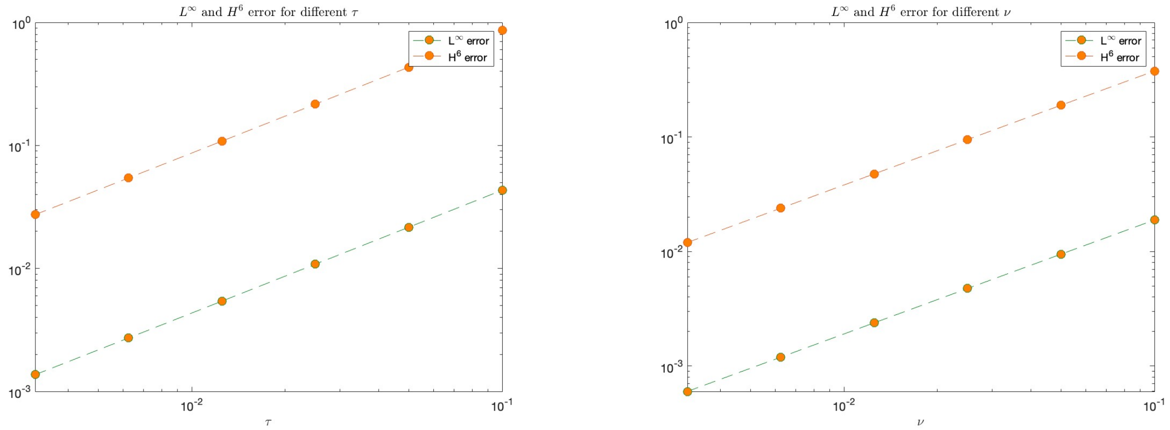


Figure 26: Navier–Stokes errors: τ-refinement at fixed ν (left) and ν-refinement at fixed τ (right).

24.5 SQG: convergence, vanishing viscosity, and vortex waltz

For dissipative SQG (56), the low-regularity exponential integrator

$$\theta^{n+1} + \tau\big(e^{-\nu\tau\Lambda^\gamma} u^n \cdot \nabla\theta^{n+1}\big) = e^{-\nu\tau\Lambda^\gamma}\theta^n, \qquad u^n = \nabla^\perp\Lambda^{-1}\theta^n \tag{60}$$

has an L^2 (and $B^0_{\infty,1}$) error $O(\tau)$ independent of ν and γ under an $H^{2+\delta}$ (respectively $B^2_{\infty,1}$) assumption. Figure 27 records errors versus τ for

$$\theta_0 = -\frac{m}{2}\Big(\cos\frac{x}{2}\Big)^m\Big(\cos\frac{y}{2}\Big)^{m-1}\sin\frac{y}{2},$$

with $m = 1.8$ and $m = 2.8$. Halving ν at fixed $T = 0.1$ produces differences of size $O(\nu)$ (Figure 28 and Table 3). A vortex-waltz pair at $\nu = 10^{-4}$ is robust from $N = 128$ to $N = 1024$ (Figure 29); the same family with $m = 0.8$ ($\theta_0 \in H^s$ only for $s < 0.3$, and $\theta_0 \notin L^\infty$) is resolution-dependent (Figure 30).

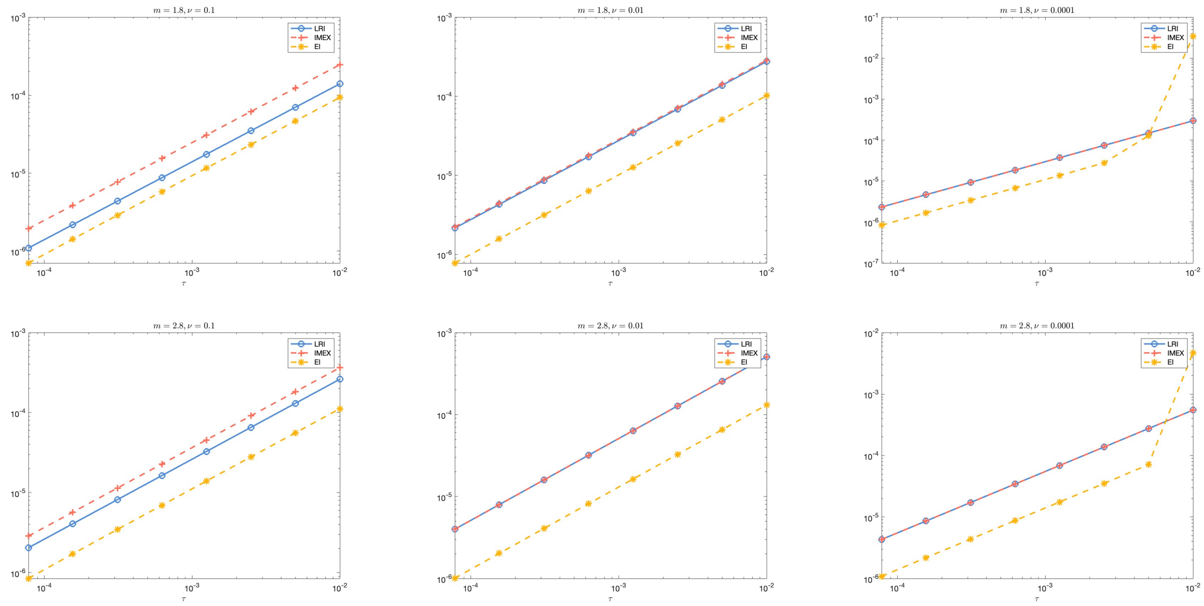


Figure 27: SQG scheme (60): errors versus τ. Top: $m = 1.8$. Bottom: $m = 2.8$. Columns: $\nu = 0.1$, 0.01, 10^{-4}.

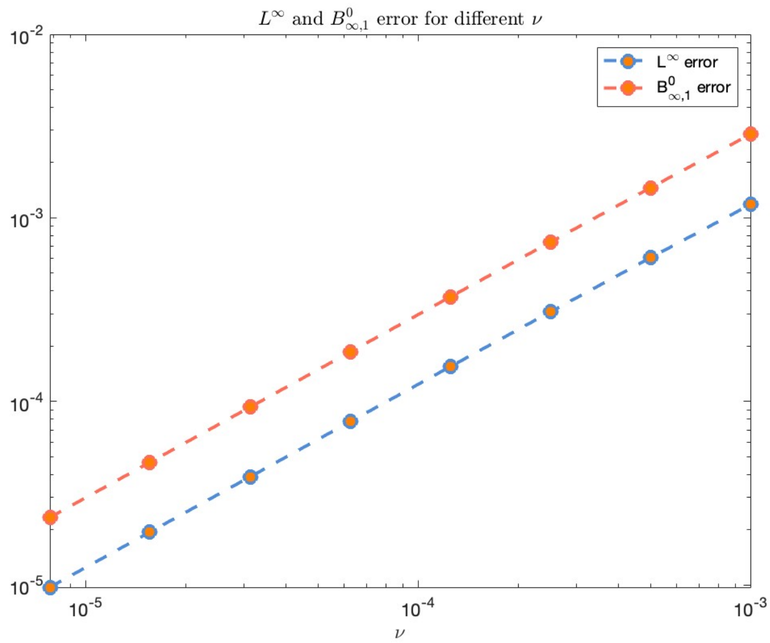


Figure 28: SQG vanishing-viscosity differences at $T = 0.1$, scaling as $O(\nu)$.

Table 3: SQG differences at $T = 0.1$ after successive halvings of ν ($m = 1.8$). Ratios ≈ 2 are consistent with an $O(\nu)$ rate.

	L^2	L^∞	$B^0_{\infty,1}$
ν	3.58×10^{-4}	5.91×10^{-4}	1.43×10^{-3}
/2	1.82×10^{-4}	3.03×10^{-4}	7.29×10^{-4}
/4	9.16×10^{-5}	1.54×10^{-4}	3.69×10^{-4}
/8	4.60×10^{-5}	7.73×10^{-5}	1.85×10^{-4}
/16	2.31×10^{-5}	3.88×10^{-5}	9.30×10^{-5}
/32	1.15×10^{-5}	1.94×10^{-5}	4.66×10^{-5}

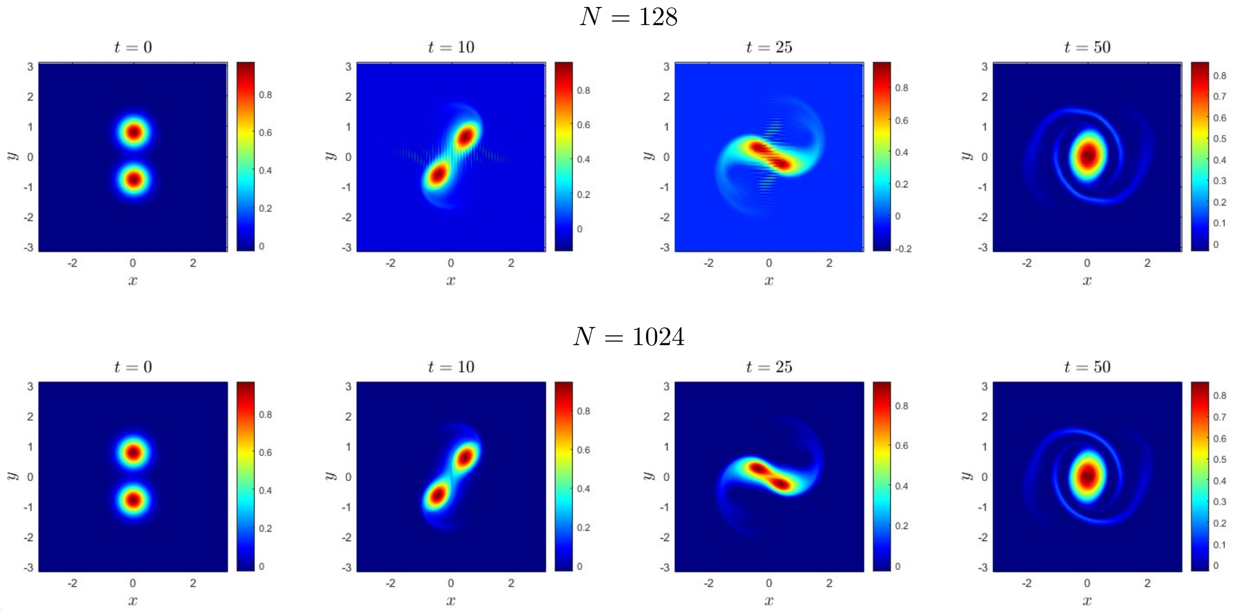


Figure 29: SQG vortex waltz, $\nu = 10^{-4}$, times $t = 0, 10, 25, 50$.

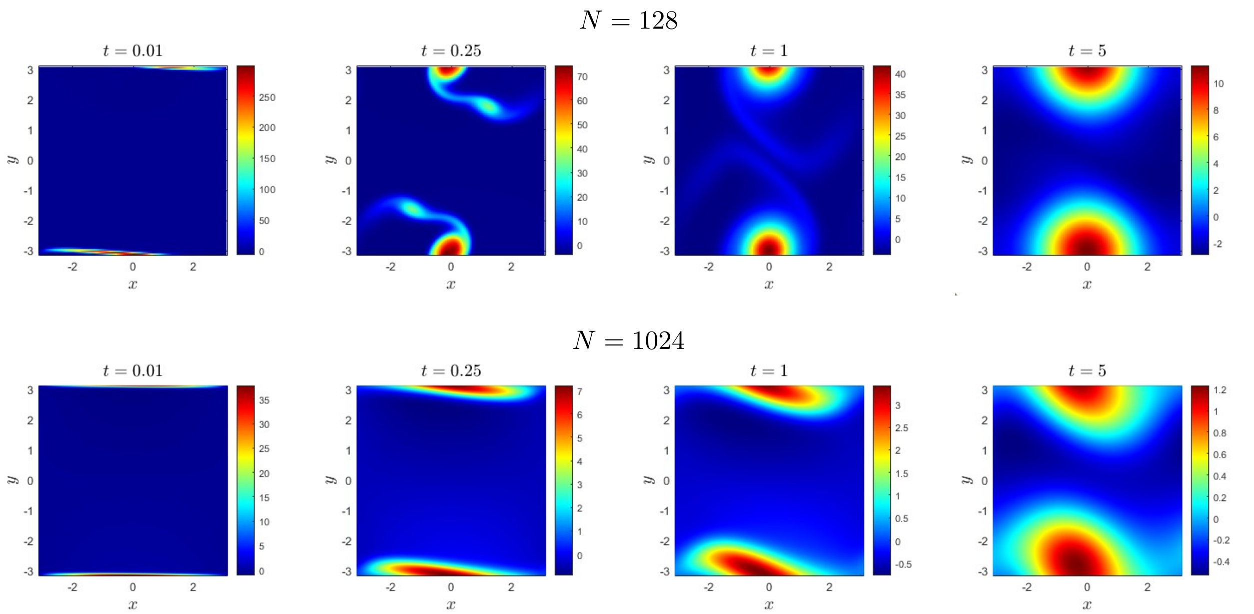


Figure 30: SQG with rough data $m = 0.8$, $\nu = 0.1$ ($\theta_0 \in H^s$ for $s < 0.3$, $\theta_0 \notin L^\infty$). The two resolutions differ, consistent with a cascade to unresolved scales.

24.6 Deep truncated FBSDE solvers

Semilinear parabolic PDEs are linked to *decoupled* FBSDEs by the nonlinear Feynman–Kac formula; quasilinear equations require a *fully coupled* FBSDE, in which the forward process X depends on (Y, Z). A naive deep BSDE solver then trains on parameter-dependent trajectories and can miss shocks in the vanishing-viscosity limit.

The truncated fictitious-play scheme (C.–Li–Xiong–Zuazua) freezes a reference path, cuts the gradient of X with respect to the network parameters, and minimizes a terminal loss plus a pathwise consistency loss. Figures 31–38 show 1D Burgers through the shock, a 100-dimensional Burgers-type equation, 2D and 100D Allen–Cahn (double-well and logarithmic potentials), and a 2D gene-flow model.

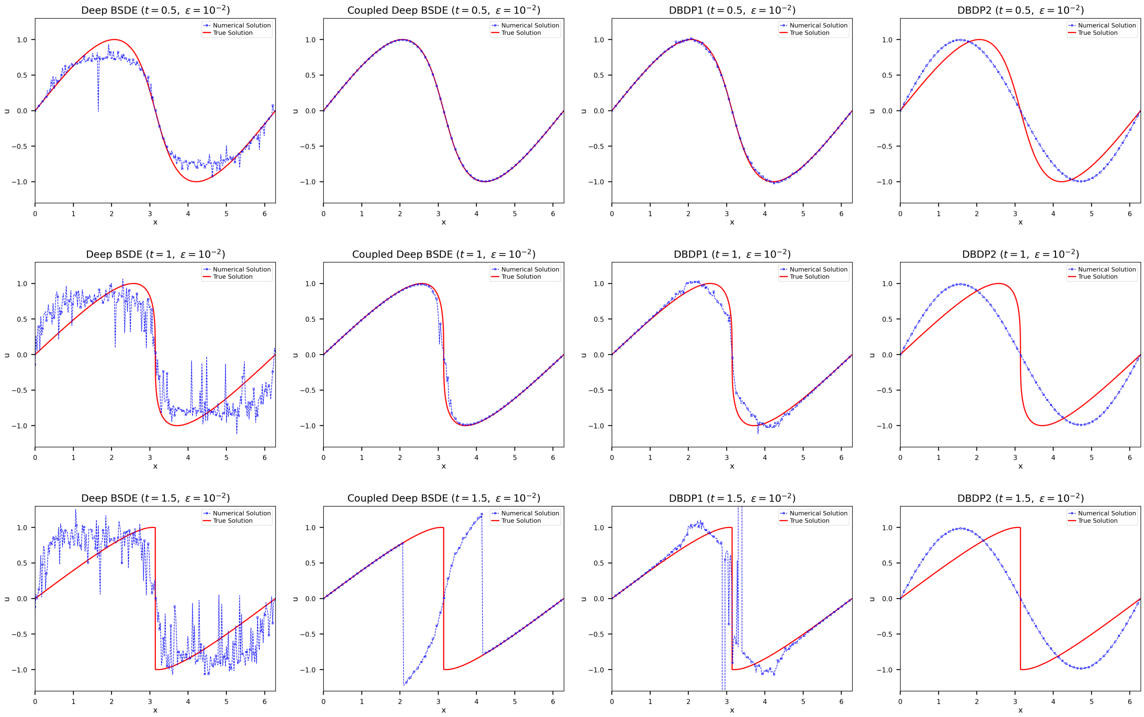


Figure 31: 1D Burgers, $\varepsilon = 10^{-2}$: deep BSDE, coupled deep BSDE, DBDP1 and DBDP2 against the true shock (rows $t = 0.5, 1.0, 1.5$).

Table 4: Deep truncated FBSDE errors for 1D Burgers. Time levels $t = 0.5, 1.0, 1.5$ use $N = 10, 20, 30$ time steps respectively. Errors remain small down to $\varepsilon = 10^{-8}$, including after the shock.

	L^2 error			L^∞ error		
ε	$t=0.5$	$t=1.0$	$t=1.5$	$t=0.5$	$t=1.0$	$t=1.5$
10^{-1}	1.06×10^{-1}	3.63×10^{-1}	4.62×10^{-1}	8.07×10^{-2}	4.99×10^{-1}	9.56×10^{-1}
10^{-2}	1.12×10^{-2}	9.64×10^{-2}	1.30×10^{-2}	8.87×10^{-3}	3.05×10^{-1}	1.94×10^{-2}
10^{-3}	1.26×10^{-3}	4.01×10^{-3}	1.48×10^{-3}	2.37×10^{-3}	1.33×10^{-2}	1.72×10^{-3}
10^{-4}	4.33×10^{-4}	5.89×10^{-4}	2.95×10^{-4}	8.96×10^{-4}	2.13×10^{-3}	4.92×10^{-4}
10^{-5}	7.97×10^{-5}	1.52×10^{-4}	7.05×10^{-5}	1.25×10^{-4}	6.18×10^{-4}	1.24×10^{-4}
10^{-6}	2.66×10^{-5}	7.48×10^{-5}	1.87×10^{-5}	5.22×10^{-5}	3.25×10^{-4}	2.56×10^{-5}
10^{-7}	2.02×10^{-5}	2.04×10^{-5}	7.58×10^{-6}	2.15×10^{-5}	7.96×10^{-5}	8.27×10^{-6}
10^{-8}	1.90×10^{-5}	1.14×10^{-5}	5.85×10^{-6}	1.52×10^{-5}	9.21×10^{-6}	6.80×10^{-6}

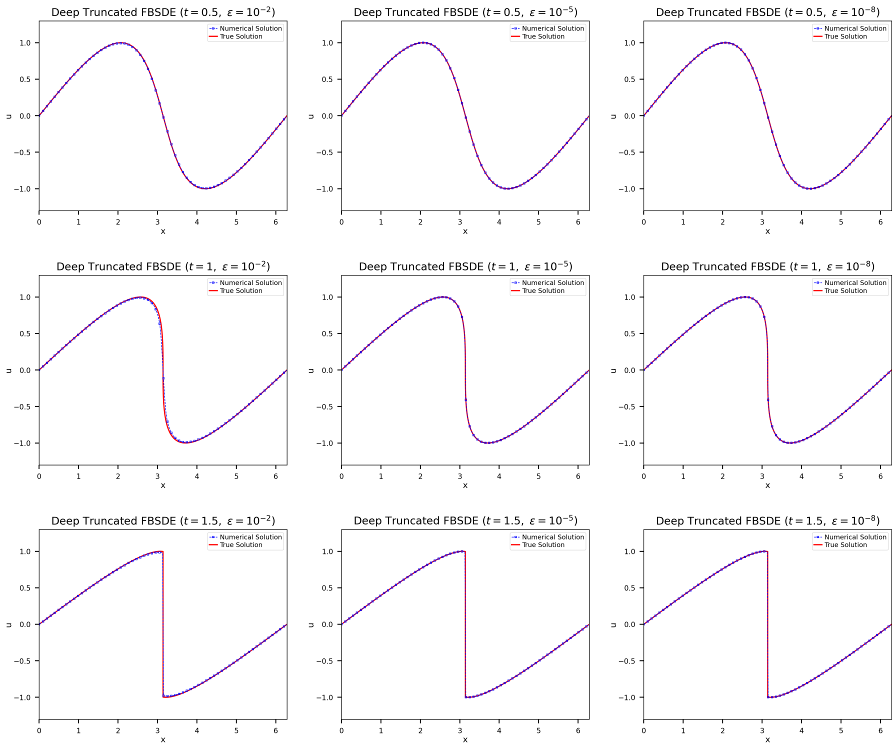


Figure 32: Deep truncated FBSDE for 1D Burgers at $\varepsilon = 10^{-2}$, 10^{-5}, 10^{-8}.

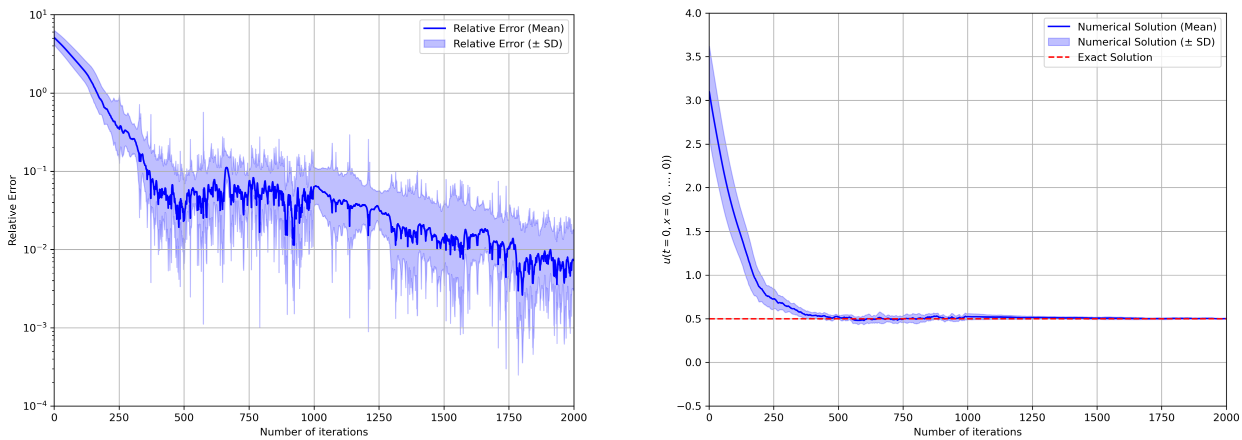


Figure 33: 100-dimensional heterogeneous Burgers-type equation: relative error (left) and the value at a monitor point (right).

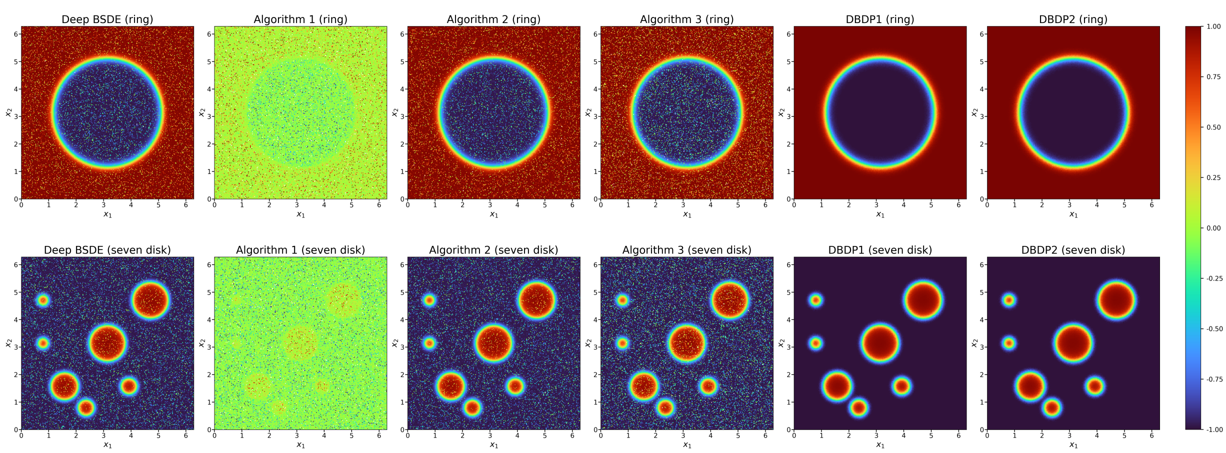


Figure 34: 2D Allen–Cahn, ring and seven-disk data: existing deep BSDE / DBDP solvers.

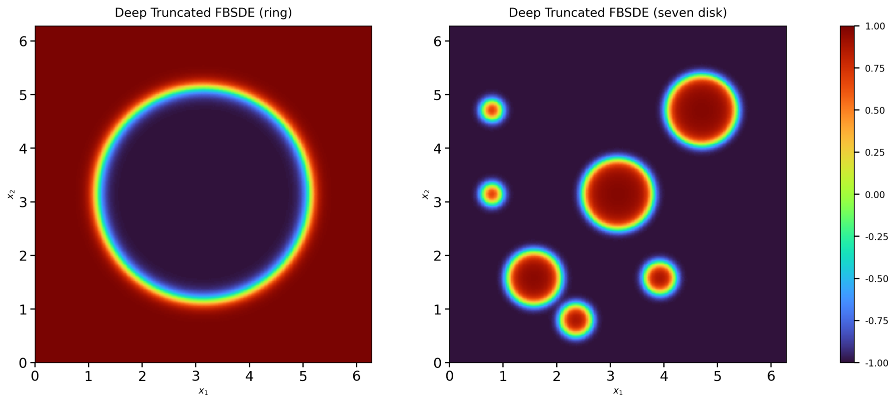


Figure 35: Same 2D Allen–Cahn tests with the truncated FBSDE solver.

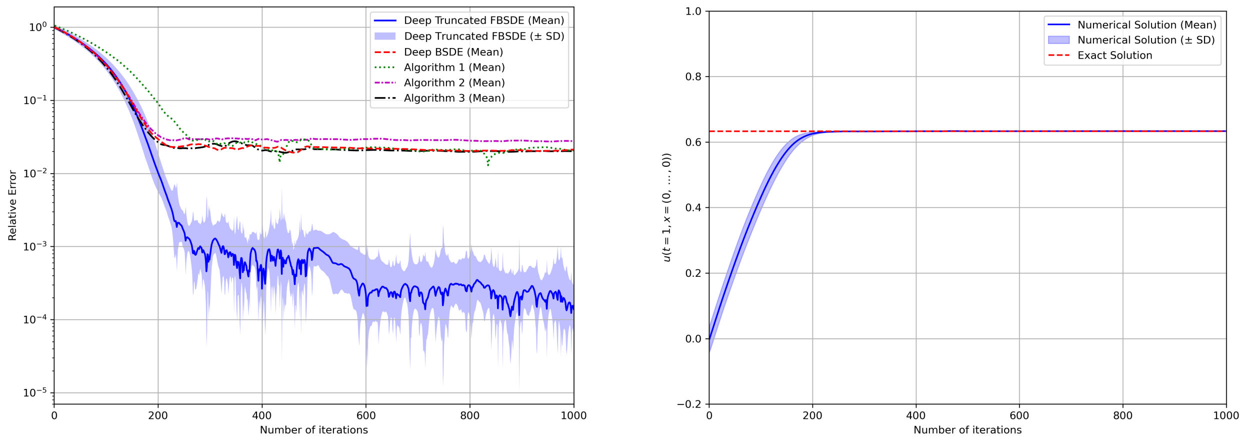


Figure 36: 100-dimensional Allen–Cahn with a double-well potential.

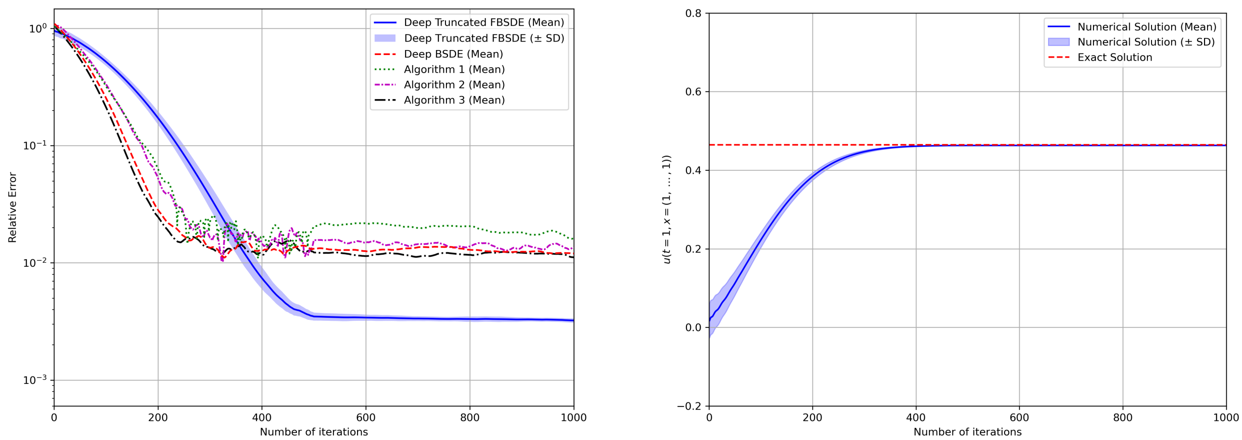


Figure 37: 100-dimensional Allen–Cahn with a logarithmic potential ($\theta_c = 1$, $\theta = \frac{1}{4}$).

Table 5: Relative errors for the 2D gene-flow model. The truncated solver degrades more slowly as T increases.

	Rel. L^2				Rel. L^∞			
Method	$T=1$	$T=3$	$T=5$	$T=7$	$T=1$	$T=3$	$T=5$	$T=7$
Deep BSDE	4.56×10^{-2}	2.59×10^{-1}	5.21×10^{-1}	6.62×10^{-1}	4.44×10^{-2}	2.09×10^{-1}	3.50×10^{-1}	4.36×10^{-1}
Alg. 1	8.81×10^{-1}	8.93×10^{-1}	9.21×10^{-1}	9.28×10^{-1}	9.98×10^{-1}	9.98×10^{-1}	9.99×10^{-1}	1.00
Alg. 2	5.07×10^{-2}	3.31×10^{-1}	6.78×10^{-1}	7.90×10^{-1}	6.14×10^{-2}	2.50×10^{-1}	4.95×10^{-1}	6.87×10^{-1}
Alg. 3	5.04×10^{-2}	2.53×10^{-1}	4.94×10^{-1}	6.35×10^{-1}	5.74×10^{-2}	1.94×10^{-1}	3.35×10^{-1}	4.39×10^{-1}
Trunc. FBSDE	8.44×10^{-3}	1.28×10^{-2}	1.83×10^{-2}	4.39×10^{-2}	2.02×10^{-2}	1.84×10^{-2}	2.44×10^{-2}	5.45×10^{-2}

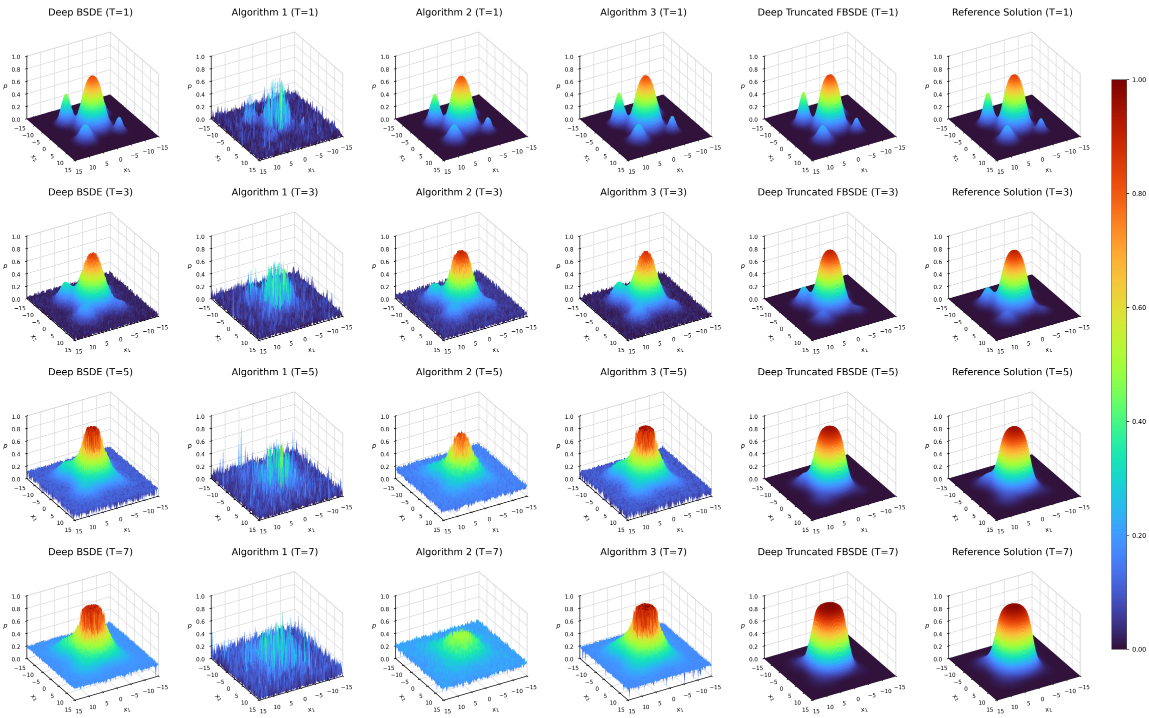


Figure 38: 2D gene-flow model at $T = 1, 3, 5, 7$: deep BSDE variants versus the truncated solver and a reference.